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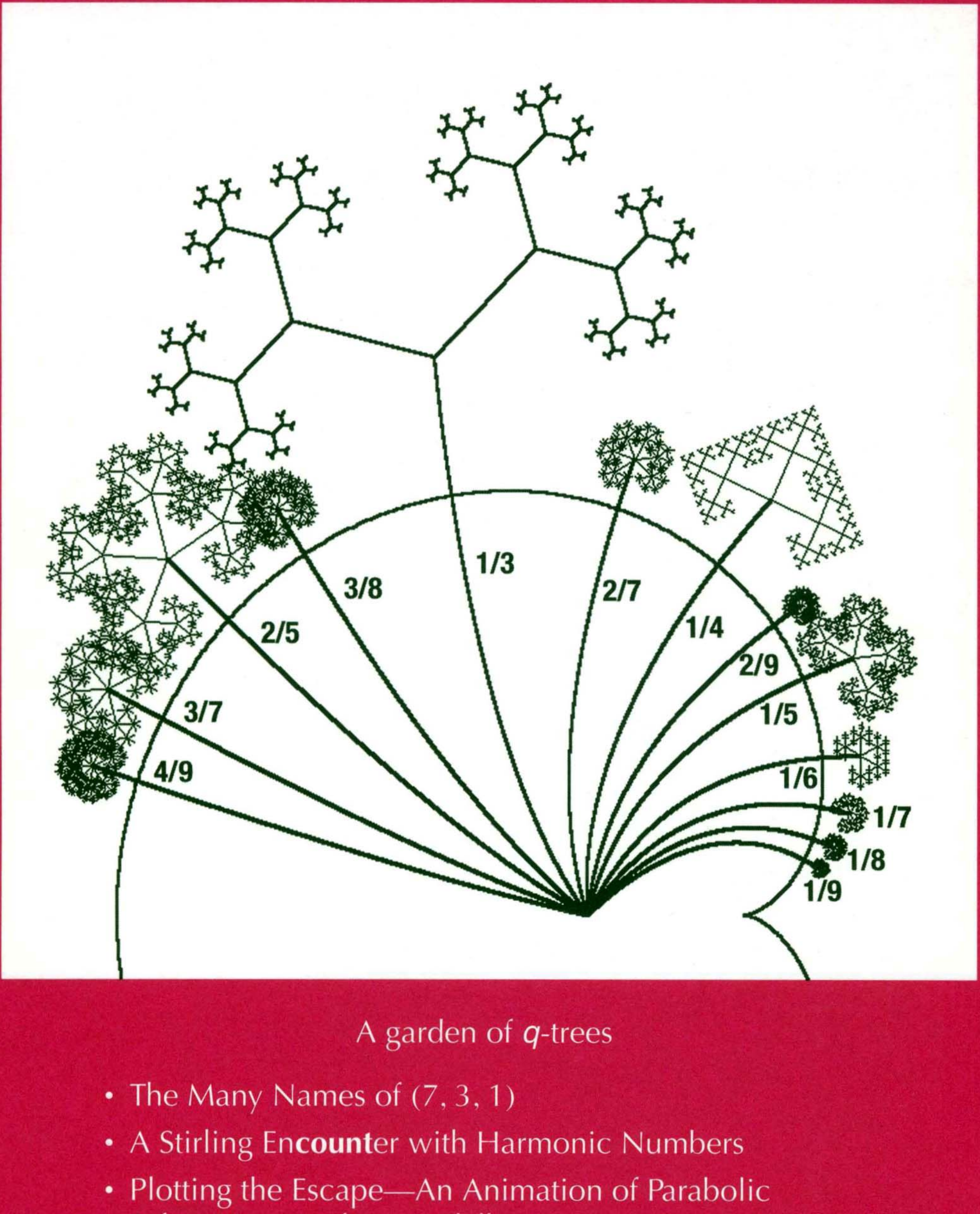
MATHEMATICS MAGAZINE



A garden of *q*-trees

- The Many Names of (7, 3, 1)
- A Stirling En**count**er with Harmonic Numbers
- Plotting the Escape—An Animation of Parabolic Bifurcations in the Mandelbrot Set

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Cover image by Anne Burns, with technical assistance from Jason Challas. This is a garden of q-trees suggested by paths of escape from the Mandelbrot set via curves of internal argument p/q. The actual construction (see Burns' article for definitions):

Starting at the origin, for $q = 3, 4, 5, \ldots, 9$, follow the path of internal argument p/q for $1 \leq p < q$, $\gcd(p, q) = 1$. Each of these paths enters the bulb $M(p, q)$ and travels to the center of the bulb where it ramifies into the $q - 1$ paths of internal argument p/q for $p = 1, \ldots, q - 1$. The process continues recursively as each of the $q - 1$ paths again ramifies into $q - 1$ new paths. For artistic reasons the paths are not drawn to scale; the size of each tree is approximately proportional to $\sin(p/q)/(q * q)$ and the recursion continues until the length of a path is less than one pixel length.

AUTHORS

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The Many Names of (7, 3, 1)

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In the world of discrete mathematics, we encounter a bewildering variety of topics with no apparent connection between them. But appearances are deceptive.

For example, combinatorics tells us about difference sets, block designs, and triple systems. Geometry leads us to finite projective planes and Latin squares. Graph theory introduces us to round–robin tournaments and map colorings, linear algebra gives us (0, 1)–matrices, and quadratic residues are among the many pearls of number theory. We meet the torus, that topological curiosity, while visiting the local doughnut shop or tubing down a river. Finally, in these fields we encounter such names as Euler, Fano, Fischer, Hadamard, Heawood, Kirkman, Singer and Steiner.

This is a story about a single object that connects all of these.

Commonly known as (7, 3, 1), it is all at once a difference set, a block design, a Steiner triple system, a finite projective plane, a complete set of orthogonal Latin squares, a doubly regular round-robin tournament, a skew-Hadamard matrix, and a graph consisting of seven mutually adjacent hexagons drawn on the torus.

We are going to investigate these connections. Along the way, we'll learn about all of these topics and just how they are tied together in one object—namely, (7, 3, 1). We'll learn about what all of those people have to do with it. We'll get to know this object quite well!

So let's find out about the many names of (7, 3, 1).

Combinatorial designs

The first place we meet (7, 3, 1) is in the set $Q_7 = \{1, 2, 4\}$. These are the nonzero perfect squares (mod 7), and their six nonzero differences, $1-2$, $1-4$, $2-4$, $2-1$, $4-1$, and $4-2$, yield each of the six distinct nonzero residues (mod 7) exactly once. Notice that Q_7 is a collection of 3 numbers mod 7, such that every nonzero integer mod 7 can be represented in exactly one way as a difference (mod 7) of distinct elements of Q_7. More generally, a (v, k, λ) *difference set* is a set S of k nonzero integers mod v such that every nonzero integer n mod v can be represented as a difference of elements of S in exactly λ different ways. Thus, Q_7 is a (7, 3, 1) difference set; from here on, we'll usually call it (7, 3, 1).

Difference sets have been the objects of a great deal of attention over the years, even before Singer constructed many families of them in his fundamental paper **[18]**. The first one students usually meet is (7, 3, 1) (or, rather, Q_7). We notice that $Q_{11} = \{1, 3, 4, 5, 9\}$ is a (11, 5, 2) difference set, since each nonzero number mod 11 can be written as a difference of elements of Q_{11} in exactly two ways (try it), and

$$Q_{47} = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 17, 18, 21, 24, 25, 27, 28, 32, 34, 36, 37, 42\}$$

is a (47, 23, 11) difference set (oh, go ahead and try it).

There's a pattern here: Q_{11} and Q_{47} are the nonzero squares mod 11 and 47, respectively, and this is no accident. It is not too tough to prove that the nonzero squares mod p form a difference set, where $p = 4n + 3$ is a prime. All we need are a few facts about numbers, but first let us set some notation. Let p be a prime number; write $[1..n]$ to mean the set $\{1, \ldots, n\}$. Write $a \equiv b \bmod p$ to mean that $a - b$ is an integral multiple of p, and call a a square mod p when there exists an x with $x^2 \equiv a \bmod p$. Let $Z_p^\times$ denote the nonzero integers mod p, let Q_p denote the nonzero squares mod p, and let $GCD(n, k)$ denote the greatest common divisor of n and k.

Here are the facts, with some (hints) about how to verify them:

- $Z_p^\times$ is a group under multiplication mod p. (The multiples of p form a subgroup of the integers, and Z_p is the resulting quotient group.)
- The squares in $Z_p^\times$ are a subgroup of $Z_p^\times$. (The squares are closed under multiplication, and $Z_p^\times$ is a finite group.)
- The product of a square and a nonsquare mod p is a nonsquare mod p, and the product of two nonsquares mod p is a square mod p. (The squares and nonsquares are the two cosets in $Z_p^\times$ mod the subgroup of squares.)
- If $p = 4n + 3$, then -1 is not a square mod p. (The key here is Lagrange's Theorem.)
- If $GCD(a, p) = 1$, then multiplication by a permutes the elements of $Z_p^\times$. (If $GCD(a, p) = 1$ and $ax \equiv ay \bmod p$, then $x \equiv y \bmod p$.)

THEOREM 1. *Let $p = 4n + 3$ be a prime. Then the nonzero squares mod p form a $(4n + 3, 2n + 1, n)$ difference set.*

Proof. For convenience, by a square (respectively, nonsquare) we will mean a member of Q_p (respectively, a member of $Z_p^\times - Q_p$). Let $x \in [1..p - 1]$. Since $x \not\equiv 0 \bmod p$, it follows that x has an inverse in $Z_p^\times$; denote this inverse by x^{-1}. (For example, if $p = 19$, then $7^{-1} = 11$ because $7 \cdot 11 = 77 \equiv 1 \bmod 19$.)

Now let R be the set of pairs of squares, that is, let $R := \{(a, b) \in [1..p - 1]: a$ and b are squares, $a \neq b\}$. We say that the pair (a, b) *represents* x if $a - b \equiv x (\bmod p)$; write $N(x)$ to mean the number of pairs in R that represent x. Define the map σ_x on ordered pairs mod p by

$$\sigma_x(a, b) = \begin{cases} (x^{-1}a, x^{-1}b), & \text{if } x \text{ is a square;} \\ (-x^{-1}b, -x^{-1}a), & \text{if } x \text{ is a nonsquare.} \end{cases}$$

For example, both 5 and 6 are squares mod 19 and 7 is a nonsquare mod 19; if $x = 7$, then we have

$$\sigma_7(5, 6) = (-7^{-1} \cdot 6, -7^{-1} \cdot 5) = (-11 \cdot 6, -11 \cdot 5) = (10, 2),$$

all arithmetic being done mod 19.

The first thing to observe is that if (a, b) represents x, then $\sigma_x(a, b)$ represents 1. For, if $a - b \equiv x (\bmod p)$, then $x^{-1}a - x^{-1}b = x^{-1}(a - b) \equiv x^{-1}x \equiv 1 (\bmod p)$. Also, $-x^{-1}b - (-x^{-1}a) \equiv x^{-1}(a - b) \equiv 1 (\bmod p)$. Now if x is a square, then $x^{-1}a$ and $x^{-1}b$ are both squares, and so $(x^{-1}a, x^{-1}b)$ represents 1. If x is a nonsquare, then $-x^{-1}$ is a square, so $-x^{-1}b$ and $-x^{-1}a$ are squares, and $(-x^{-1}b, -x^{-1}a)$ represents 1. Thus, every representation of x leads to a representation of 1.

On the other hand, $\sigma_{x^{-1}}$ is an inverse map of σ_x, so that if (c, d) represents 1, then $\sigma_{x^{-1}}(c, d)$ represents x. Thus, for all x, every representation of 1 leads to a representation of x.

We conclude that $N(x) = N(1)$ for all $x \in [1..p - 1]$, and so every $x \in [1..p - 1]$ has the same number of representations. This lets us count R: it contains $N(1) \cdot (p - 1) = N(1) \cdot (4n + 2)$ pairs.

Finally, since multiplication by -1 permutes $Z_p^\times$ and exchanges the squares and nonsquares, there are $2n+1 = (p-1)/2$ squares mod p; hence, R contains $(2n+1)2n = \frac{p-1}{2}(\frac{p-1}{2}-1)$ pairs (there are no pairs (a, a)). Equating these two values for the size of R, we see that $N(1) = (2n+1)2n/(4n+2) = n$. Hence, every nonzero integer mod $p = 4n+3$ is represented n times by a difference of nonzero squares mod p—that is, the nonzero squares form a $(4n+3, 2n+1, n)$ difference set. ∎

There are many other classes of difference sets. For example, $B_{13} = \{0, 1, 3, 9\}$ is a difference set with $v = 13$, and $B_{37} = \{1, 7, 9, 10, 12, 16, 26, 33, 34\}$ is a difference set with $v = 37$. (As an exercise, verify these statements and in so doing, determine k and λ for each. Question: Except for 0, B_{13} looks like the powers of 3 mod 13; is there a similar pattern for B_{37}?)

As we have seen, $\{1, 2, 4\}$ is a (7, 3, 1) difference set. But so is any additive shift $\{1+n, 2+n, 4+n\}$ (mod 7) of $\{1, 2, 4\}$. Consider all seven of these sets together; writing *abc* for the set $\{a, b, c\}$, we have

$$124, 235, 346, 450, 561, 602, \text{ and } 013. \tag{1}$$

This is well illustrated by rotating the triangle in FIGURE 1 counterclockwise within its circumscribing circle:

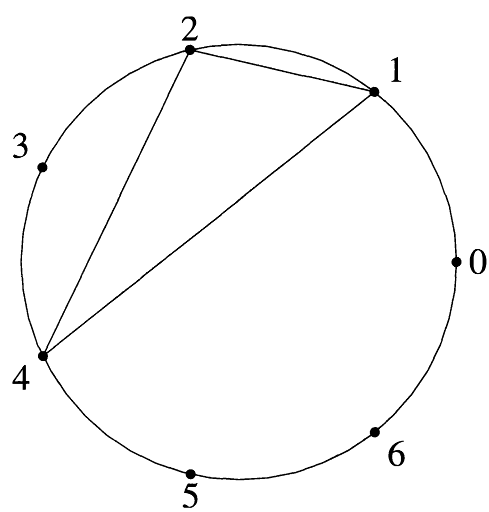

Figure 1 The (7, 3, 1) difference set

Notice that for these 7 sets (or *blocks*), whose elements are taken from a 7-element set, namely [0..6], each element appears in 3 blocks, each block has 3 elements, and each pair of elements appears together in exactly one block. The difference sets in this section give rise to some special classes of what are called *block designs*, and some more names for (7, 3, 1). So let's talk about block designs.

A *balanced incomplete block design*, or BIBD with parameters b, v, r, k, and λ is an arrangement of b blocks, taken from a set of v objects (known for historical reasons as *varieties*), such that every variety appears in exactly r blocks, every block contains exactly k varieties, and every pair of varieties appears together in exactly λ blocks. Such an arrangement is also called a (b, v, r, k, λ) design. Thus, (7, 3, 1) is a (7, 7, 3, 3, 1) design. Block designs appeared in connection with the eminent British statistician R. A. Fisher's work on the statistical design of agricultural experiments ([**7**], [**8**]), and the first comprehensive mathematical study of the field was due to R. C. Bose [**2**]. Now, the five parameters are by no means independent, for it turns out that $bk = vr$ and $r(k-1) = \lambda(v-1)$ (exercise: prove it). Hence, a (b, v, r, k, λ) design is really a $(\lambda v(v-1)/(k(k-1)), v, \lambda(v-1)/(k-1), k, \lambda)$ design. If $b = v$ (and hence $r = k$), the design is said to be *symmetric*; thus, (7, 3, 1) is a (7, 3, 1) symmetric design.

The familiar 3×3 magic square (see FIGURE 2, on the right), in which the rows, columns, and main diagonals all add up to 15, is the source of another block design.

1	2	3		4	9	2
4	5	6		3	5	7
7	8	9		8	1	6

Figure 2 Generating a design from a 3×3 magic square

Here's how it works: first, arrange the integers from 1 to 9 in that order in a 3×3 grid—see FIGURE 2, on the left. Allowing diagonals to wrap when they reach the edge of the grid (as if there were another copy of the grid next door) yields twelve 3-element sets: three rows, three columns, and six diagonals. Thus, we have 9 objects arranged in 12 blocks with each object in four blocks, each block containing three objects and each pair of objects in one block–in short, a (12, 9, 4, 3, 1) design. Here are the blocks:

$$123, 456, 789, 147, 258, 369, 168, 249, 357, 159, 267, 348.$$

Now we have already seen that the seven additive shifts (mod 7) of our (7, 3, 1) difference set form the blocks of a (7, 3, 1) symmetric design. In addition, the eleven additive shifts (mod 11) of our $(11, 5, 2)$ difference are the blocks of a symmetric $(11, 5, 2)$ design (see for yourself). It turns out that this is the case in general, and we can prove it.

THEOREM 2. *Let $D = \{x_1, x_2, \ldots, x_k\}$ be a (v, k, λ) difference set. Let $B_i := \{x_1 + i, \ldots, x_k + i\}$ where addition is mod v. Then the v sets $B_0, \ldots, B_{v-1}$ are the blocks of a (v, k, λ) symmetric design.*

Proof. By definition, there are v blocks and v varieties. By construction, there are k varieties in each block. In addition, since $y = x_j + (y - x_j)$ for $1 \le j \le k$, each $y \in [0..v-1]$ appears in blocks $B_{y-x_1}, \ldots, B_{y-x_k}$. Hence each variety appears in k blocks. Finally, let $y, z \in [0..v-1]$; then y and z are in B_t if and only if $t = y - x_i = z - x_j$ for distinct $i, j \in [1..k]$. This happens if and only if $y - z = x_i - x_j$; since D is a (v, k, λ) difference set, this happens for exactly λ pairs (x_i, x_j). Thus, there are exactly λ values of t for which $t = y - x_i = z - x_j$ for distinct $i, j \in [1..k]$, and for these values, y and z appear together in a block. ∎

You may wonder whether the converse of this theorem is also true—that is, does every (v, k, λ) symmetric design give rise to a (v, k, λ) difference set? Interesting question: we'll come back to it later.

Finally, a class of block designs that has attracted considerable interest over the years is the one for which $k = 3$ and $\lambda = 1$. Such a design is called a *Steiner triple system* on v varieties, or STS(v) for short. Since (7, 3, 1) certainly has $k = 3$, it is also an STS—in fact, the smallest nontrivial Steiner triple system. Now if an STS(v) exists, then $v \equiv 1$ or 3 (mod 6), which follows from the fact that a $(b, v, r, 3, 1)$ design is really a $(\lambda v(v-1)/(3 \cdot 2), v, (v-1)/2, 3, 1)$ design. (You can work this one out!)

Steiner posed the problem of showing that triple systems exist for all such $v \ge 3$, but he did not solve it. In fact, the problem had been solved more than a decade earlier by the Reverend Thomas A. Kirkman [**11**]—see Doyen's survey article [**5**] for a great deal more about these triple systems. (Perhaps they should be renamed in honor of Kirkman, about whom more later.)

Steiner triple systems turn up in some unlikely places, such as subfield diagrams in algebraic number theory—but that's another story.

Since the 3×3 magic square is a (12, 9, 4, 3, 1) design, it is also an STS(9). But it is more than that: the words *magic square* suggest some connection with geometry. Curiously enough, it turns out that (7, 3, 1) has geometric connections as well. So, let's talk about finite geometries.

Finite geometries

The 3×3 magic square we met in the previous section (FIGURE 2 on the left) is an example of a finite geometry. For, if by a *line*, we mean a set of points—not necessarily connected, straight, or infinite—then the 3×3 magic square obeys some fairly simple rules:

(1) Each pair of points lies on a unique line.

(2) Each pair of lines intersects in at most one point.

(3) There exist four points with no three on a line.

The first two rules are reminiscent of Euclidean plane geometry, and the third ensures that the object at hand is nontrivial. Arrangements that satisfy these three rules are called *finite affine planes*, and the number of points on each line is called the *order* of the plane. Thus, the 3×3 magic square gives rise to a finite affine plane (FAP) of order 3.

We cannot draw a picture of this in the plane without two pairs of lines crossing unnecessarily, but we can draw it on a torus—the surface of a doughnut—by wrapping the diagonals. See FIGURE 3, where solid lines represent the lines of this finite plane, and dotted lines indicate wrapping on the torus.

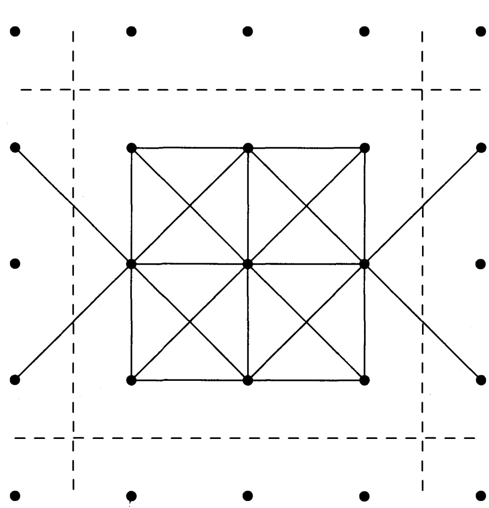

Figure 3 The finite affine plane of order 3

Here's a question: What is the smallest possible finite plane? We need at least four points with no three on a line; if we call the points A, B, C, and D, then the six lines AB, AC, AD, BC, BD, and CD form a perfectly good finite plane. Now in our affine planes, some of the lines intersect and some of them do not.

But what if we insisted that the plane be *projective*, that is, that every pair of lines have a unique point in common? What is the smallest possible finite projective plane (FPP)?

Let's add some points to the plane above. Clearly, AB and CD must meet in some point X, AC and BD meet in some point Y, and AD and BC meet in some point Z. If $X = Y$, then A, B, and X are on a line, and B, D, and $X(= Y)$ are on a line. But B and X determine a unique line, so that A, B, and D are on a line—contrary to assumption. Hence, $X \neq Y$. For the same reason, $X \neq Z$ and $Y \neq Z$.

We know that an FPP contains at least seven points, and so far, it contains the six lines ABX, CDX, ACY, BDY, ADZ, and BCZ. There must be a line through X and Y. To keep things small, we add the line XYZ (represented by the circle on the left in FIGURE 4); then each pair of lines intersects in a unique point. The resulting seven-point FPP is known as the *Fano plane*; here it is on the left in FIGURE 4:

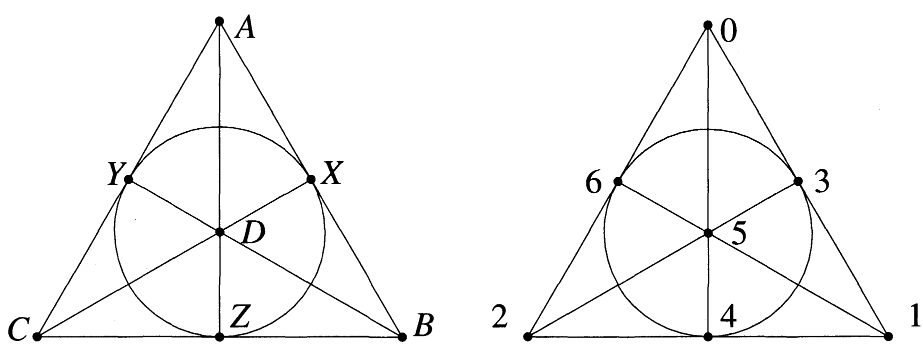

Figure 4 The Fano plane

Notice that the Fano plane has seven points and seven lines; each line contains three points, each point is on three lines and each pair of points is on exactly one line. Sound suspiciously familiar? It should, for if we replace A, B, C, D, X, Y, and Z with 0, 1, 2, 5, 3, 6, and 4, respectively, the lines look like this:

$$124, \quad 235, \quad 346, \quad 450, \quad 561, \quad 602, \quad \text{and} \quad 013,$$

and $(7, 3, 1)$ has reappeared—on the right in FIGURE 4—as the Fano plane. This configuration was named for G. Fano, who described it in 1892 [**6**]. In another twist of fate, however, he was anticipated by Woolhouse in 1844 [**20**] and, yes, by Kirkman in 1850 [**12**].

More generally, a *finite projective plane of order n*, abbreviated FPP(n), is an FPP containing $n^2 + n + 1$ points and $n^2 + n + 1$ lines, such that every point is on $n + 1$ lines, every line contains $n + 1$ points, and every pair of points is on a unique line. Thus, an FPP(n) is an $(n^2 + n + 1, n + 1, 1)$ symmetric design; conversely, every $(v, k, 1)$ design is a finite projective plane of order $n = k - 1$ with $v = n^2 + n + 1$.

One of the major unsolved problems in combinatorics is determining the values of n for which an FPP(n) exists. Their existence is equivalent to the existence of certain families of designs called Latin squares, designs that got mixed up with one of the most famous false conjectures in the history of mathematics. So let's talk briefly about them now.

A *Latin square of order n* is an $n \times n$ array with entries from the set $[1..n]$ such that each element of $[1..n]$ appears in each row and each column of the array exactly once. Two Latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ of order n are said to be *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}) are distinct. (A good reference for the general subject of Latin squares is a text by Dénes and Keedwell [**4**].) Here are three Latin squares of order 4; you can check that (a) they are Latin squares and (b) they are orthogonal in pairs:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

It turns out that the existence of this trio of pairwise orthogonal Latin squares of size 4 is equivalent to the existence of a finite projective plane of order 4; in fact, this is true in general:

THEOREM 3. *Let n be an integer greater than 1. Then there exists a finite projective plane of order n if and only if there exists a set of $n - 1$ Latin squares of size n that are pairwise orthogonal.*

An outline of the proof is given in a text by Roberts [**16**]—who also shows that if n is a prime power, then there exists a set of $n - 1$ Latin squares of size n that are pairwise orthogonal. Hence, FPP's exist for orders 2, 3, 4, 5, 7, 8, and 9. In particular, (7, 3, 1) is a FPP(2) and so there must be a corresponding set of $n - 1$ pairwise orthogonal Latin squares of size $n = 2$. And what is that set? You can figure it out, or I'll tell you later.

So 2, 3, 4, 5, 7, 8, and 9 are all just fine. But what about order 6?

The great Leonhard Euler wondered about 6, too. That prodigious mathematical mind from the eighteenth century made a study of Latin squares, showed how to construct a pair of size n if n is not of the form $4k + 2$, and saw immediately that it is impossible to construct a pair of orthogonal Latin squares of size 2 (try it). He then attempted to construct a pair of size 6; failing to do so, he made the following bold conjecture:

EULER'S CONJECTURE (1782). For each nonnegative integer k, there does not exist a pair of orthogonal Latin squares of size $4k + 2$.

For over 100 years, nothing happened. Then, in 1900, G. Tarry wrote two papers [**19**] proving that Euler was right about 6. But as Bose, Shrikhande, and Parker [**3**] showed in 1960, he was spectacularly wrong for all other values of $4k + 2$ greater than 6:

THEOREM 4. *There exists a pair of orthogonal Latin squares of order n for all $n > 6$.*

It is now known (the details are in a nice survey by Lam [**13**]) that an FPP(10) does not exist; the smallest unknown case is for $n = 12$. So we see that (7, 3, 1) is connected with one of the rare instances in which Euler was almost totally wrong!

It happens that (7, 3, 1) has connections, not only with combinatorics, but also with that other major branch of discrete mathematics: graph theory. So let's talk about graph theory.

Graph theory connections

A *graph* G is a set of points (or vertices) $V(G)$ together with a set of lines (or edges) $E(G)$ joining some (perhaps all, perhaps none) of the points. We say that the vertices a and b are *adjacent* if the edge ab is in the graph. A graph is *complete* if there is an edge between every pair of points. Graphs can be *directed*, that is, each edge is assigned a direction: we write (a, b) if there is an edge directed from a to b. Graphs model an amazing number of problems, from simple word puzzles (the Wolf-Goat-Cabbage problem) and games (Tic–Tac–Toe and chess) to such complex systems as the continental power grid and the internet. Directed graphs can model the play of teams in a league, and that is where (7, 3, 1) comes in.

In many sports leagues, each team plays every other team exactly once, and there are no ties. We can model this scenario with a graph as follows. The teams are the vertices, and for each pair of teams u and v, include the edge (u, v) directed from u to v if u beats v, and include the edge (v, u) if v beats u. We call such a graph a *(round–robin) tournament*; thus, a tournament is a complete graph with a direction assigned to each edge. (A good reference on tournaments is Moon's book [**14**].) The score of a vertex u is the number of edges (u, v) in the tournament. A tournament is called *transitive* if every team has a different score, and *regular* if every team has the same score. Naturally, the higher the score, the higher the team's rank. Thus, in a transitive tournament, the scores determine the ranking unambiguously, and in a regular tournament, the scores don't give any information. (A transitive tournament is so named because it has the property that if u beats v and v beats w, then u beats w.)

Now, the high schools of Auburn, Blacksburg, Christiansburg, EastMont, Giles, Newman, and Radford make up the Riverside League. In most years, one or two schools dominate, but last year the results of the league's round-robin play were quite different:

Team	**Victories Over**
Auburn	Blacksburg, Giles, Radford
Blacksburg	Christiansburg, Giles, Newman
Christiansburg	Auburn, Newman, Radford
EastMont	Auburn, Blacksburg, Christiansburg
Giles	Christiansburg, EastMont, Radford
Newman	Auburn, Giles, EastMont
Radford	Blacksburg, EastMont, Newman

Each team had the same score of 3, so this was a regular tournament. But the league was even more balanced than that: each pair of teams was victorious over exactly one common opponent. (Such a tournament is called *doubly regular*.) Let us look at this a little more carefully, assigning numbers to the teams as follows: $A = 3$, $B = 0$, $C = 1$, $E = 6$, $G = 4$, $N = 2$, and $R = 5$. If we now rewrite the results in numerical order, the table of victories begins to look somewhat familiar:

Team	Victories Over
0	1, 2, 4
1	2, 3, 5
2	3, 4, 6
3	4, 5, 0
4	5, 6, 1
5	6, 0, 2
6	0, 1, 3

Our old friend (7, 3, 1) has reappeared: the sets of teams defeated by each member of the league are the blocks of a (7, 3, 1) symmetric design, and the teams defeated by Team 0 form a (7, 3, 1) difference set.

Of course, this is no accident—and we can prove it.

THEOREM 5. *Let $p = 4n + 3$ be a prime. Define the tournament T by $V(T) = [0..p - 1]$ and $E(T) = \{(x, x + r)$: r is a square mod $p\}$. Then T is a doubly regular tournament with $4n + 3$ vertices, in which every vertex has a score of $2n + 1$ and every pair of vertices defeats n common opponents.*

Proof. Since there are $(p - 1)/2 = 2n + 1$ squares mod p, each vertex has a score of $2n + 1$. Now let x, y be distinct vertices. Then (x, z) and (y, z) are both edges of T if and only if there exist distinct squares r and s such that $z - x = r$ and $z - y = s$. Hence, the number of such z is equal to the number of pairs of distinct squares r, s such that $r - s = x - y$. But the squares form a $(4n + 3, 2n + 1, n)$ difference set, and so the nonzero number $x - y$ can be written as a difference $r - s$ in exactly n distinct ways. Hence, there are n vertices z such that both (x, z) and (y, z) are edges of T, that is, T is doubly regular. ∎

In short, (7, 3, 1) is a doubly regular tournament.

Now, the adjacency matrix of a tournament T on v vertices $x_1, \ldots, x_v$ is a $v \times v$ matrix $A = [A_{ij}]$ such that $A_{ij} = 1$ if there is an edge from x_i to x_j, and $A_{ij} = 0$ otherwise. Thus, the (7, 3, 1) doubly regular tournament has the following adjacency matrix A:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

If we replace all the 0s in A with -1s and border the resulting matrix top and left with a row and column of 1s, we obtain the above matrix H. If we multiply H by its transpose H^T, it turns out that $HH^T = 8I$, where I is the 8×8 identity matrix. An $n \times n$ matrix H of 0s and 1s for which $HH^T = nI$ is called a *Hadamard matrix* of order n—not a property enjoyed by very many $(-1, 1)$-matrices. It turns out that the adjacency matrix of every doubly regular tournament can be transformed into a skew–Hadamard matrix (one for which $H + H^T = 2I$), and every skew–Hadamard matrix gives rise to a doubly regular tournament [**15**].

Hadamard matrices are very useful in constructing error–correcting codes and other combinatorial designs. You can show that if H is a Hadamard matrix of order $n > 1$, then either $n = 2$ or $n \equiv 0 \bmod 4$. Whether the converse is true is an unsolved problem.

More graph theory connections

The final two $(7, 3, 1)$ connections turned up in my course in Galois theory, that beautiful subject in which Évariste Galois (1811–1832) related the roots of polynomials to number fields and finite groups. One basic idea is that if $p(x)$ is a polynomial with rational coefficients, then there is a smallest subfield $L(p)$ of the complex numbers $\mathbb{C}$ containing both the rationals $\mathbb{Q}$ and all the roots of $p(x)$. This is the *splitting field* of p over $\mathbb{Q}$. If $a, b, \ldots \in \mathbb{C}$ and if K is a subfield of $\mathbb{C}$, write $K(a, b, \ldots)$ to mean the smallest subfield of $\mathbb{C}$ containing K and $a, b, \ldots$.

For the polynomial $p(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)$, it turns out that $L(p) = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Now this field contains, besides itself and $\mathbb{Q}$, fourteen other subfields which—to my great delight—form a $(7, 3, 1)$ design. For this particular incarnation of $(7, 3, 1)$, the varieties are the seven quadratic subfields $\mathbb{Q}(\sqrt{d})$, where $d \in \{2, 3, 5, 6, 10, 15, 30\}$, and the blocks are the seven biquadratic subfields $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ (where $d_1 d_2$ is not a perfect square). See for yourself:

Biquadratic Field	**Contains $\mathbb{Q}(\sqrt{d})$ for these d**
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	2, 3, 6
$\mathbb{Q}(\sqrt{3}, \sqrt{5})$	3, 5, 15
$\mathbb{Q}(\sqrt{5}, \sqrt{6})$	5, 6, 30
$\mathbb{Q}(\sqrt{6}, \sqrt{15})$	6, 15, 10
$\mathbb{Q}(\sqrt{15}, \sqrt{30})$	15, 30, 2
$\mathbb{Q}(\sqrt{30}, \sqrt{10})$	30, 10, 3
$\mathbb{Q}(\sqrt{10}, \sqrt{2})$	10, 2, 5

In short, $(7, 3, 1)$ appears in the subfields of the splitting field of a polynomial.

But there's one more connection. If we join two subfields of L by a line if one contains the other and there is no intermediate field, we get the lattice of subfields of L. For our field $L(p)$, if we do this and ignore $L(p)$ and $\mathbb{Q}$, the left side of FIGURE 5 shows what we get.

This is the $(3, 6)$ cage: every vertex has degree 3, the shortest cycle has length 6, and no graph with fewer vertices has these properties. The $(3, 6)$ cage cannot be drawn in the plane without edges crossing. It can, however, be drawn on the torus, and the right side of FIGURE 5 shows what *that* looks like.

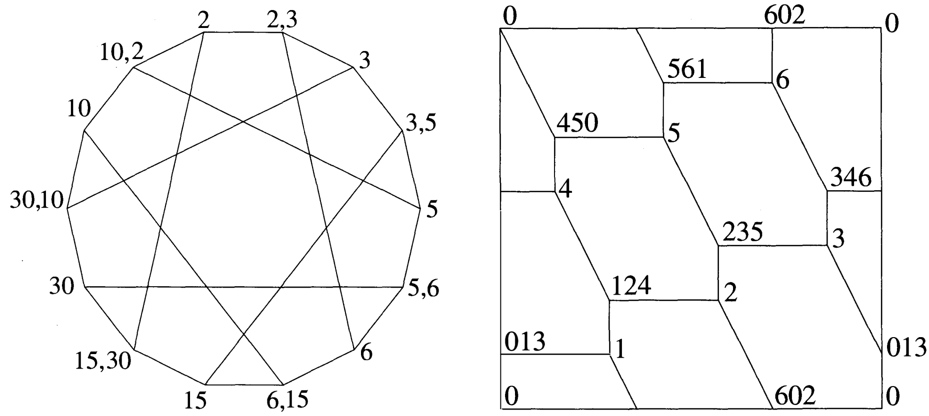

Figure 5 The Heawood graph

This toroidal embedding is what gives the graph its common name: the Heawood graph. For, in 1890, Percy J. Heawood proved that every graph that can be drawn on the torus without edges crossing requires at most seven colors to color its regions with neighboring regions having different colors. The Heawood graph was his example of a toroidal graph requiring seven colors, for it consists of seven mutually adjacent hexagons.

With that, we see how (7, 3, 1) has a connection with Heawood's 7–Color Theorem for toroidal graphs, and with the Heawood graph—just one more of the many names of (7, 3, 1).

Questions

Does (7, 3, 1) have any other names? Yes, it does. There is a combinatorial design called a (3, 4, 7, 2) *configuration of size 14*; it consists of fourteen 3-element subsets of [1..7], no more than two of which lie in a common 4-element subset of [1..7]. The incidence graph of this design (two 3-sets are joined if and only if they lie in a common 4-set) is the Heawood graph. To find more names, Richard Guy's paper **[9]** is an excellent place to start.

Where can I find out more about difference sets? One of the best places to begin is with H. J. Ryser's beautifully written book **[17]**, which will take you a fair way into the subject. Two others are the more recent book by Beth, Jungnickel, and Lenz **[1]** and Marshall Hall's classic **[10]**, both of which will take you as far as you want to go into the subject. All three of these also give good introductions to the other combinatorial designs talked about here.

Do all (v, k, λ) symmetric designs give rise to (v, k, λ) difference sets? In fact, they don't—but the smallest example is $(v, k, \lambda) = (25, 9, 3)$. Exercise: Find it.

How did Tarry prove that there does not exist a pair of orthogonal Latin squares of size 6? Brute force. He used symmetry arguments to reduce the number of cases to about six thousand—then eliminated them, one by one. (Don't try this at home.)

What is the complete set of orthogonal Latin squares that corresponds to (7, 3, 1)? We know that (7, 3, 1) is also an FPP of order $n = 2$, so the corresponding set of $n - 1 = 1$ orthogonal Latin square(s) of size 2 is just

$$A = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right\}.$$

Did the Reverend Thomas Kirkman ever get credit for anything? Yes, he did. In 1850, he posed what is known as Kirkman's Schoolgirls Problem. Fifteen schoolgirls take daily walks, arranged in five rows of three each; arrange the girls so that for seven consecutive days, no girl is in a row with the same companion more than once. The solution to this problem is a particularly interesting Steiner triple system on 15 varieties, with parameters (35, 15, 7, 3, 1)—but that's another story.

Speaking of Steiner triple systems: are there Steiner quadruple systems? In fact, the idea generalizes to a Steiner system $S(k, m, n)$, which is a collection C of m-element subsets of an n—element set B, such that every k-element subset of B is contained in exactly one of the sets in C. An $S(2, 3, n)$ is a Steiner triple system, and an $S(2, n+1, n^2+n+1)$ is a finite projective plane. Not too many of these are known with $k > 3$. One of these is known as $S(5, 8, 24)$; it has many extraordinary properties, as well as connections with—ahh, but *that's* another story!

REFERENCES

1. T. Beth, D. Jungnickel, and H. Lenz, *Design Theory, 2nd Edition*, Cambridge University Press, Cambridge, 1999.
2. R. C.Bose, On the construction of balanced incomplete block designs, *Ann. Eugenics* **9** (1939), 353–399.
3. R. C. Bose, S. Shrikhande, and E. T. Parker, Further results on the construction of sets of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canad. J. Math.* **12** (1960), 189–203.
4. Jozsef Dénes and Donald Keedwell, *Latin Squares and their Applications*, Academic Press, New York, 1974.
5. Jean Doyen, Recent developments in the theory of Steiner systems, *Atti dei Conv. Lincei* **17** (1976), 277–285.
6. G. Fano, Sui postulati fondamentali della geometria proiettiva, *Giorn. Mat.* **30** (1892), 114–124.
7. R. A. Fisher, *The Design of Experiments*, Oliver and Boyd, Edinburgh, 1935.
8. R. A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugenics* **10** (1940), 52–57.
9. Richard K. Guy, The unity of combinatorics, in C. J. Colburn and E. S. Mahmoodian (eds) *Combinatorics Advances*, Kluwer, 1995, 129–159.
10. Marshall Hall, Jr., *Combinatorial Theory* (9th edition), Blaisdell Publishing Company, Waltham, MA, 1967.
11. T. A. Kirkman, On a problem in combinations, *Camb. Dublin Math. J.* **2** (1847), 191–204.
12. T. A. Kirkman, Note on an unanswered prize question, *Camb. Dublin Math. J.* **5** (1850), 255–262.
13. Clement W. H. Lam, The search for a finite projective plane of order 10, *Amer. Math. Monthly* **98** (1991), 305–318.
14. J. W. Moon, *Topics on Tournaments*, Holt, Rinehart and Winston, New York, 1968.
15. K. B. Reid and E. Brown, Doubly regular tournaments are equivalent to skew–Hadamard matrices, *J. Combinatorial Theory, Series A* **12** (1972), 332–338.
16. Fred S. Roberts, *Applied Combinatorics*, Prentice–Hall, 1984.
17. H. J. Ryser, *Combinatorial Mathematics*, Carus Mathematical Monograph No. 14, Mathematical Association of America, Washington, DC, 1963.
18. J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc.* **43** (1938), 377–385.
19. G. Tarry, Le Problème de 36 officiers, *C. R. Assoc. Fr. Avance. Sci. Nat* **1** (1900), 122–123; **2** (1901), 170–203.
20. W. S. B. Woolhouse, Prize question 1733, *Lady's and Gentleman's Diary*, 1844.

A Stirling Encounter with Harmonic Numbers

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Harmonic numbers are defined to be partial sums of the harmonic series. For $n \geq 1$, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The first five harmonic numbers are $H_1 = 1$, $H_2 = 3/2$, $H_3 = 11/6$, $H_4 = 25/12$, $H_5 = 137/60$. For convenience we define $H_0 = 0$. Since the harmonic series diverges, H_n can get arbitrarily large, although it does so quite slowly. For instance, $H_{1,000,000} \approx 14.39$.

Harmonic numbers even appear in real life. If you stack 2-inch long playing cards to overhang the edge of a table as far as possible, the maximum distance that n cards can hang off the edge of the table is H_n [**5**]. For example, 4 cards can be stacked to extend past the table by just over 2 inches, since $H_4 = 25/12$.

Harmonic numbers satisfy many interesting properties. For nonnegative integers n and m, we list some identities below:

$$\sum_{k=1}^{n-1} H_k = nH_n - n. \tag{1}$$

$$\sum_{k=m}^{n-1} \binom{k}{m} H_k = \binom{n}{m+1}\left(H_n - \frac{1}{m+1}\right). \tag{2}$$

$$\sum_{k=m}^{n-1} \binom{k}{m} \frac{1}{n-k} = \binom{n}{m}\left(H_n - H_m\right). \tag{3}$$

Although all of these identities can be proved by algebraic methods (see [**5**]), the presence of binomial coefficients suggests that these identities can also be proved *combinatorially.* A combinatorial proof is a counting question, which when answered two different ways, yields both sides of the identity. Combinatorial proofs often provide intuitive and concrete explanations where algebraic proofs may not. For example

$$\sum_{k=1}^{n-1} k \cdot k! = n! - 1$$

is a standard exercise in mathematical induction. But to a combinatorialist this identity counts permutations in two different ways. The right side counts the number of ways to arrange the numbers 1 through n, excluding the natural arrangement $1\ 2\ 3\ \ldots\ n$. The left side counts the same quantity by *conditioning* on the first number that is not in its natural position: for $1 \leq k \leq n-1$, how many arrangements have $n-k$ as the first number to differ from its natural position? Such an arrangement begins as $1\ 2\ 3\ \ldots\ n-k-1$ followed by one of k numbers from the set $\{n-k+1, n-k+2, \ldots, n\}$. The remaining k numbers (now including the number $n-k$) can be arranged $k!$ ways. Thus there are $k \cdot k!$ ways for $n-k$ to be the first misplaced number. Summing over all feasible values of k yields the left side of the identity.

Although H_n is never an integer for $n > 1$ [**5**], it can be expressed as a rational number whose numerator and denominator have combinatorial significance. Specifically, for $n \geq 0$ we can always write

$$H_n = \frac{p_n}{n!} \tag{4}$$

as a (typically nonreduced) fraction where p_n is a nonnegative integer.

Now $p_0 = H_0 = 0$. For $n \geq 1$, $H_n = H_{n-1} + 1/n$ leads to

$$\frac{p_n}{n!} = \frac{p_{n-1}}{(n-1)!} + \frac{1}{n} = \frac{np_{n-1} + (n-1)!}{n!}.$$

Hence for $n \geq 1$,

$$p_n = np_{n-1} + (n-1)! \tag{5}$$

The combinatorial interpretation of these numbers is the topic of the next section.

Stirling numbers

For integers $n \geq k \geq 1$, let $\left[{n \atop k}\right]$ denote the number of permutations of n elements with exactly k cycles. Equivalently $\left[{n \atop k}\right]$ counts the number of ways for n distinct people to sit around k identical circular tables, where no tables are allowed to be empty. $\left[{n \atop k}\right]$ is called the (unsigned) *Stirling number* of the first kind. As an example, $\left[{3 \atop 2}\right] = 3$ since one person must sit alone at a table and the other two have one way to sit at the other table. We denote these permutations by $(1)(23)$, $(13)(2)$, and $(12)(3)$.

We can compute the numbers $\left[{n \atop k}\right]$ recursively. From their definition, we see that for $n \geq 1$,

$$\left[{n \atop 1}\right] = (n-1)!,$$

since the arrangement $(a_1a_2a_3 \ldots a_n)$ is the same as arrangements $(a_2a_3 \ldots a_na_1)$ and $(a_3a_4 \ldots a_1a_2)$ and so on. Now for $k \geq 2$, we will see that

$$\left[{n+1 \atop k}\right] = \left[{n \atop k-1}\right] + n\left[{n \atop k}\right]. \tag{6}$$

On the left, we are directly counting the number of ways to seat $n+1$ people around k circular tables. On the right we count the same thing while conditioning on what happens to person $n+1$. If $n+1$ is to be alone at a table, then the remaining n people can be arranged around $k-1$ tables in $\left[{n \atop k-1}\right]$ ways. If $n+1$ is not to be alone, then we first arrange 1 through n around k tables (there are $\left[{n \atop k}\right]$ ways to do this); for each of these configurations, we insert person $n+1$ to the right of any of the n already-seated people. This gives us $n\left[{n \atop k}\right]$ different permutations where $n+1$ is not alone. Summing gives equation (6).

Notice that when $k=2$, equation (6) becomes

$$\left[{n+1 \atop 2}\right] = n\left[{n \atop 2}\right] + (n-1)!, \tag{7}$$

which is the same as recurrence (5) with $p_n = \left[{n+1 \atop 2}\right]$. Since $p_1 = 1 = \left[{2 \atop 2}\right]$, it follows that for all $n \geq 1$, $p_n = \left[{n+1 \atop 2}\right]$. Combining with the definition of p_n in (4) gives

THEOREM 1. *For $n \geq 1$,*

$$H_n = \frac{1}{n!}\left[{n+1 \atop 2}\right].$$

Next we show how to count Theorem 1 *directly*—without relying on a recurrence. First we set some notational conventions. Let $\mathcal{T}_n$ denote the set of arrangements of the numbers 1 through n into two disjoint, nonempty cycles. Thus $|\mathcal{T}_n| = \left[{n \atop 2}\right]$. We always write our cycles with the smallest element first, and list the cycles in increasing order according to the first element. For example, $\mathcal{T}_9$ includes the permutation (185274)(396), but not (195)(2487)(36) nor (123)(4567)(8)(9). By our convention, the cycle containing 1 is always written first; consequently we call it the *left cycle*. The remaining cycle is called the *right cycle.* All permutations in $\mathcal{T}_n$ are of the form $(a_1a_2\ldots a_j)(a_{j+1}\ldots a_n)$, where $1 \leq j \leq n-1$, $a_1 = 1$, and a_{j+1} is the smallest element of the right cycle.

For a purely combinatorial proof of Theorem 1 that does not rely on a recursion, we ask, for $1 \leq k \leq n$, how many permutations of $\mathcal{T}_{n+1}$ have exactly k elements in the right cycle? To create such a permutation, first choose k elements from $\{2, \ldots, n+1\}$ ($\binom{n}{k}$ ways), arrange these elements in the right cycle ($(k-1)!$ ways), then arrange the remaining $n-k$ elements in the left cycle following the number 1 ($(n-k)!$ ways). Hence there are $\binom{n}{k}(k-1)!\,(n-k)! = n!/k$ permutations of $\mathcal{T}_{n+1}$ with k elements in the right cycle. Since $\mathcal{T}_{n+1}$ has $\left[{n+1 \atop 2}\right]$ permutations, it follows that

$$\left[{n+1 \atop 2}\right] = \sum_{k=1}^{n} \frac{n!}{k} = n!H_n,$$

as desired.

Another way to prove Theorem 1 is to show that for $2 \leq r \leq n+1$, there are $\frac{n!}{r-1}$ permutations in $\mathcal{T}_{n+1}$ that have r as the minimum element of the right cycle.

Here, the permutations being counted have the form $(1\ldots)(r\ldots)$ where elements 1 through $r-1$ all appear in the left cycle, and elements $r+1$ through $n+1$ can go in either cycle. To count this, arrange elements 1 through $r-1$ into the left cycle, listing element 1 first; there are $(r-2)!$ ways to do this. Place element r into the right cycle. Now we insert elements $r+1$ through $n+1$, one at a time, each immediately to the *right* of an already placed element. In this way, elements 1 and r remain first (and smallest) in their cycles. Specifically, the element $r+1$ can go to the right of any of the elements 1 through r. Next, $r+2$ can go to the right of any of the elements 1 through $r+1$. Continuing in this way, the number of ways to insert elements $r+1$ through $n+1$ is $r(r+1)(r+2)\cdots n = n!/(r-1)!$. This process creates a permutation in $\mathcal{T}_{n+1}$ with r as the smallest element in the right cycle. Thus, there are

$$(r-2)!\,\frac{n!}{(r-1)!} = \frac{n!}{r-1}$$

such permutations. Since $\mathcal{T}_{n+1}$ has $\left[{n+1 \atop 2}\right]$ permutations, and every permutation in $\mathcal{T}_{n+1}$ must have some smallest integer r in the right cycle, where $2 \le r \le n+1$, we get

$$\left[{n+1 \atop 2}\right] = \sum_{r=2}^{n+1} \frac{n!}{r-1} = n! \sum_{k=1}^{n} \frac{1}{k} = n!\, H_n.$$

An alternate way to see that $n!/(r-1)$ counts permutations of the form $(1 \cdots)(r \cdots)$ is to list the numbers 1 through $n+1$ in any order with the provision that 1 be listed first. There are $n!$ ways to do this. We then convert our list $1\ a_2\ a_3 \cdots\ r \cdots\ a_{n+1}$ to the permutation $(1\ a_2\ a_3 \cdots)(r \cdots a_{n+1})$ by inserting parentheses. This permutation satisfies our conditions if and only if the number r is listed to the right of elements $2, 3, \ldots, r-1$. This has probability $1/(r-1)$ since any of the elements $2, 3, \ldots, r$ have the same chance of being listed last among them. Hence the number of permutations that satisfy our conditions is $n!/(r-1)$.

Algebraic connection The Stirling numbers can also be defined as coefficients in the expansion of the rising factorial function [**3**]:

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{m=1}^{n} \left[{n \atop m}\right] x^m. \tag{8}$$

Using this definition, Theorem 1 can be derived algebraically by computing the x^2 coefficient of $x(x+1)(x+2)\cdots(x+n)$.

To show that this algebraic definition of Stirling numbers is equivalent to the combinatorial definition, one typically proves that both satisfy the same initial conditions and recurrence relation. However, a more direct correspondence exists [**1**], which we illustrate with an example.

By the algebraic definition, the Stirling number $\left[{10 \atop 3}\right]$ is the coefficient of x^3 in the expansion $x(x+1)(x+2)\cdots(x+9)$. The combinatorial definition says $\left[{10 \atop 3}\right]$ counts the number of ways that elements $0, 1, 2, \ldots, 9$ can sit around 3 identical circular tables. Why are these definitions the same? Each term of the x^3 coefficient is a product of seven numbers chosen from among 1 through 9. Surely this must be counting something. What is a term like $1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9$ counting?

As illustrated in FIGURE 1, this counts the number of ways elements 0 through 9 can seat themselves around 3 identical tables where the smallest elements of the tables are the "missing" numbers 0, 4, and 7. To see this, we pre-seat numbers 0, 4, 7 then seat the remaining numbers one at a time in increasing order. The number 1 has just one option—sit next to 0. The number 2 then has two options—sit to the right of 0 or sit to the right of 1. The number 3 now has three options—sit to the right of 0 or 1 or 2. The number 4 is already seated. Now number 5 has five options—sit to the right of 0 or 1 or 2 or 3 or 4, and so on. A general combinatorial proof of equation (8) can also be done by the preceding (or should that be "pre-seating"?) argument.

With this understanding of the interactions between harmonic and Stirling numbers, we now provide combinatorial explanations of other harmonic identities.

Recounting harmonic identities

In this section, we convert identities (1), (2), and (3) into statements about Stirling numbers and explain them combinatorially. We view each identity as a story of a counting problem waiting to be told. Each side of the identity *recounts* the story in a different, but accurate way. Both of our combinatorial proofs of Theorem 1 were

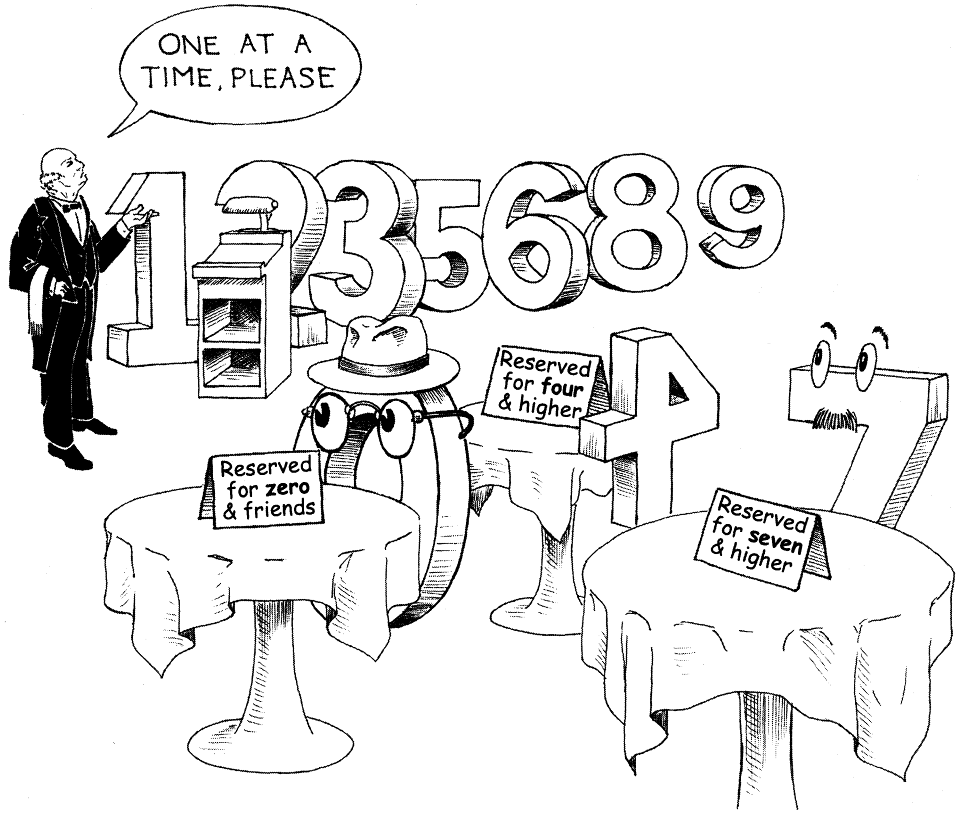

Figure 1 How many ways can the numbers 1, 2, 3, 5, 6, 8, 9 seat themselves around these tables?

obtained by partitioning the set $\mathcal{T}_{n+1}$ according to the size of the right cycle or the minimum element of the right cycle, respectively. In what follows, we shall transform harmonic equations (1), (2) and (3) into three Stirling number identities, each with $\left[{n \atop 2}\right]$ on the left-hand side. The right-hand sides will be combinatorially explained by partitioning $\mathcal{T}_n$ according to the location of element 2, the largest of the last t elements, or the *neighborhood* of the elements 1 through m. Our first identity, after applying Theorem 1, and re-indexing ($n := n - 1$) gives us

IDENTITY 1. *For* $n \geq 2$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! + \sum_{k=1}^{n-2} \frac{(n-2)!}{k!} \begin{bmatrix} k+1 \\ 2 \end{bmatrix}.$$

To prove this combinatorially, we note that the left side of the identity, $\left[{n \atop 2}\right]$, counts the number of permutations in $\mathcal{T}_n$. On the right, we know from our second combinatorial proof of Theorem 1, that $(n-1)!$ counts the number of permutations in $\mathcal{T}_n$ where the number 2 appears in the right cycle. It remains to show that the summation above counts the number of permutations in $\mathcal{T}_n$ where 2 is in the left cycle. Any such permutation has the form

$$(1\ a_1\ a_2 \cdots a_{n-2-k}\ 2\ b_1\ b_2 \cdots b_{j-1})(b_j \cdots b_k),$$

for some $1 \leq k \leq n-2$ and $1 \leq j \leq k$. We assert that the number of these permutations with exactly k terms to the right of 2 is given by the kth term of the sum.

To see this, select $a_1, a_2, \ldots, a_{n-2-k}$ from the set $\{3, \ldots, n\}$ in any of $(n-2)!/k!$ ways. From the unchosen elements, there are $\left[{k+1 \atop 2}\right]$ ways to create two nonempty cycles of the form $(2\ b_1 \ldots b_{j-1})(b_j \ldots b_k)$ where $1 \leq j \leq k$. Multiplying the two counts gives the kth term of the sum as the number of permutations in $\mathcal{T}_n$ with exactly k terms to the right of 2, as was to be shown.

We apply a different combinatorial strategy to prove the more general equation (2), which, after applying Theorem 1 and re-indexing ($n := n-1$, $m := t-1$, and $k := k-2$), gives us

IDENTITY 2. *For* $1 \leq t \leq n-1$

$$\left[{n \atop 2}\right] = \frac{(n-1)!}{t} + t \sum_{k=t+1}^{n} \left[{k-1 \atop 2}\right] \frac{(n-1-t)!}{(k-1-t)!}.$$

The combinatorial proof of this identity requires a new interpretation of $(n-1)!/t$. For $1 \leq t \leq n-1$, we define the *last t elements* of $(1a_2 \cdots a_j)(a_{j+1} \cdots a_n)$ to be the elements $a_n, a_{n-1}, \ldots a_{n+1-t}$, even if some of them are in the left cycle. For example, the last 5 elements of (185274)(396) are 6, 9, 3, 4, and 7.

We claim that for $1 \leq t \leq n-1$, the number of permutations in $\mathcal{T}_n$ where the largest of the last t elements is alone in the right cycle is $(n-1)!/t$. Here, we are counting permutations of the form $(1a_2 \ldots a_{n-1})(a_n)$, where a_n is the largest of $\{a_{n+1-t}, a_{n+2-t}, \ldots, a_{n-1}, a_n\}$. Among all $(n-1)!$ permutations of this form, the largest of the last t elements is equally likely to be anywhere among the last t positions. Hence $(n-1)!/t$ of them have the largest of the last t elements in the last position.

Next we claim that for $1 \leq t \leq n-1$, the number of permutations in $\mathcal{T}_n$ where the largest of the last t elements is *not* alone in the right cycle is the summation in Identity 2.

To see this, we count the number of such permutations where the largest of the last t elements is equal to k. Since the number 1 is not listed among the last t elements, we have $t+1 \leq k \leq n$. To construct such a permutation, we begin by arranging numbers 1 through $k-1$ into two cycles. Then insert the number k to the right of any of the last t elements. There are $\left[{k-1 \atop 2}\right]t$ ways to do this. The right cycle contains at least one element less than k, so k is not alone in the right cycle (and could even be in the left cycle). So that k remains the largest among the last t elements, we insert elements $k+1$ through n, one at a time, to the right of any but the last t elements. There are $(k-t)(k+1-t)\cdots(n-1-t) = (n-1-t)!/(k-1-t)!$ ways to do this. Multiplying the two counts give the kth term of the sum as the number of permutations where the largest of the last t elements equals k, and it is not alone in the right cycle; summing over all possible values of k, we count all such permutations. Since for any permutation in $\mathcal{T}_n$, the largest of the last t elements is either alone in the last cycle, or it isn't, and this establishes Identity 2.

Notice that when $t = 1$, Identity 2 simplifies to Identity 1. When $t = n-1$, Identity 2 essentially simplifies to equation (7).

For our final identity, we convert equation (3) to Stirling numbers using Theorem 1 and re-indexing ($n := n-1$, $m := m-1$, and $k := t-1$). This gives us

IDENTITY 3. *For* $1 \leq m \leq n$

$$\left[{n \atop 2}\right] = \left[{m \atop 2}\right] \frac{(n-1)!}{(m-1)!} + \sum_{t=m}^{n-1} \binom{t-1}{m-1} \frac{(m-1)!\,(n-m)!}{(n-t)}.$$

To prove this identity combinatorially, we condition on whether numbers 1 through m all appear in the left cycle. First we claim that for $1 \leq m \leq n$, the first term on the right in the identity counts the number of permutations in $\mathcal{T}_n$ that do not have elements $1, 2, \ldots m$ all in the left cycle: For these permutations, the elements 1 through m can be arranged into two cycles in $\left[{m \atop 2}\right]$ ways. Insert the remaining elements $m+1$ through n, one at a time, to the right of any existing element, finding that there are $m(m+1)\cdots(n-1) = (n-1)!/(m-1)!$ ways to insert these elements. Multiplying the two counts gives the first term of the right-hand side.

To complete the proof, we must show that the summation on the right counts the number of permutations in $\mathcal{T}_n$ where elements 1 through m are all in the left cycle. To see this, we claim that for $m \leq t \leq n-1$, the summand counts the permutations described above with exactly t elements in the left cycle and $n-t$ elements in the right cycle. To create such a permutation, we first place the number 1 at the front of the left cycle. Now choose $m-1$ of the remaining $t-1$ spots in the left cycle to be assigned the elements $\{2, \ldots, m\}$. There are $\binom{t-1}{m-1}$ ways to select these $m-1$ spots and $(m-1)!$ ways to arrange elements $2, \ldots, m-1$ in those spots. For example, to guarantee that elements 1, 2, 3, 4 appear in the left cycle of FIGURE 2, we select three of the five open spots in which to arrange 2, 3, 4. The insertion of 5, 6, 7, 8, 9 remains. Now there are $(n-m)!$ ways to arrange elements $m+1$ through n in the remaining spots, but only one out of $n-t$ of them will put the smallest element of the right cycle at the front of the right cycle. Hence, elements $m+1$ through n can be arranged in $(n-m)!/(n-t)$ legal ways. Multiplying gives the number of ways to satisfy our conditions for a given t, and the total is given by the desired summation.

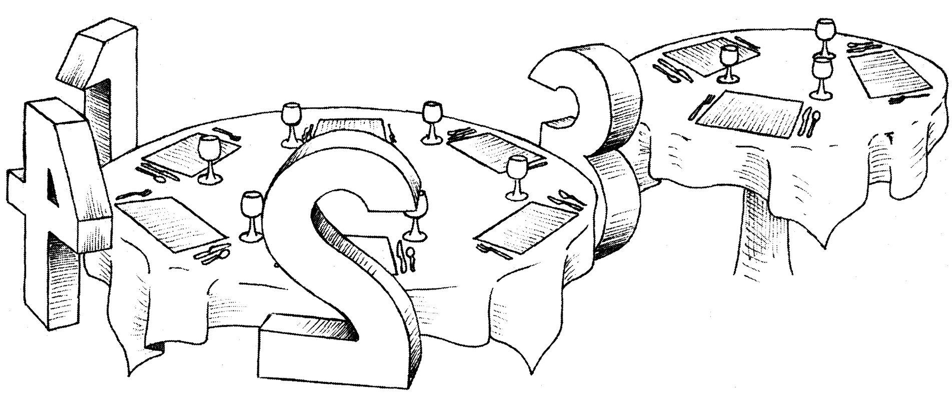

Figure 2 In $\mathcal{T}_9$, a permutation with 1, 2, 3, 4 in a left cycle containing exactly six elements is created by first selecting three of the five open spots, and then arranging 2, 3, 4 in them. Subsequently, 5, 6, 7, 8, 9 will be arranged in the remaining spots.

We have already noted that harmonic numbers arise in real life. A further occurrence arises in calculating the average number of cycles in a permutation of n elements. Specifically,

THEOREM 2. *On average, a permutation of n elements has H_n cycles.*

There are $n!$ permutations of n elements, of which $\left[{n \atop k}\right]$ have k cycles. Consequently, Theorem 2 says

$$\frac{\sum_{k=1}^{n} k\left[{n \atop k}\right]}{n!} = H_n,$$

or equivalently, by Theorem 1,

IDENTITY 4. *For* $n \geq 1$,

$$\sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}.$$

The left side counts the number of permutations of $\{1, \ldots, n\}$ with an arbitrary number of cycles, where one of the cycles is distinguished in some way. For example $\underline{(1284)}(365)(79)$, $(1284)\underline{(365)}(79)$, and $(1284)(365)\underline{(79)}$ are three different arrangements with $k = 3$. The right side counts the number of permutations of $\{0, 1, \ldots, n\}$ with exactly two cycles. It remains to describe a one-to-one correspondence between these two sets of objects. Can you deduce the correspondence between the following three examples?

$$\underline{(1284)}(365)(79) \Longleftrightarrow (079365)(1284)$$

$$(1284)\underline{(365)}(79) \Longleftrightarrow (0791284)(365)$$

$$(1284)(365)\underline{(79)} \Longleftrightarrow (03651284)(79)$$

In general, we transform the permutation with n elements

$$(C_k)(C_{k-1}) \cdots (C_{j+1})\underline{(C_j)}(C_{j-1}) \cdots (C_2)(C_1)$$

into

$$(0\ C_1\ C_2 \cdots C_{j-1}\ C_{j+1} \cdots C_{k-1}\ C_k)(C_j).$$

The process is easily reversed. Given $(0\ a_1 \cdots a_{n-j})(b_1 \cdots b_j)$ in $\mathcal{T}_{n+1}$, the right cycle becomes the distinguished cycle $\underline{(b_1 \cdots b_j)}$. The distinguished cycle is then inserted among the cycles $C_{k-1}, \ldots C_2, C_1$, which are generated one at a time as follows: C_1 (the rightmost cycle) begins with a_1 followed by a_2 and so on until we encounter a number a_i that is less than a_1. Assuming such an a_i exists (that is, $a_1 \neq 1$), begin cycle C_2 with a_i and repeat the procedure, starting a new cycle every time we encounter a new smallest element. The resulting cycles (after inserting the distinguished one in its proper place) will be a permutation of n elements written in our standard notation. Hence we have a one-to-one correspondence between the sets counted on both sides of Identity 4.

Notice that by distinguishing exactly m of the cycles above, the procedure above can be easily modified to prove the more general

$$\sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}.$$

Likewise by distinguishing an arbitrary number of cycles, the same kind of procedure results in

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} 2^k = (n+1)!\,.$$

Beyond harmonic numbers

We have only scratched the surface of how combinatorics can offer new insights about harmonic numbers. Other combinatorial approaches to harmonic identities are presented by Preston [**6**]. We leave the reader with a challenge: A *hyperharmonic number*

$H_n^{(k)}$ is defined as follows: Let $H_n^{(1)} = H_n$ and for $k > 1$, define $H_n^{(k)} = \sum_{i=1}^{n} H_i^{(k-1)}$. Now consider the following generalization of identity (1) from *The Book of Numbers* by Conway and Guy [**4**]:

$$H_n^{(k)} = \binom{n+k-1}{k-1}(H_{n+k-1} - H_{k-1}).$$

Such an identity strongly suggests that there must be a combinatorial interpretation of hyperharmonic numbers as well. And indeed there is one [**2**]. You can count on it!

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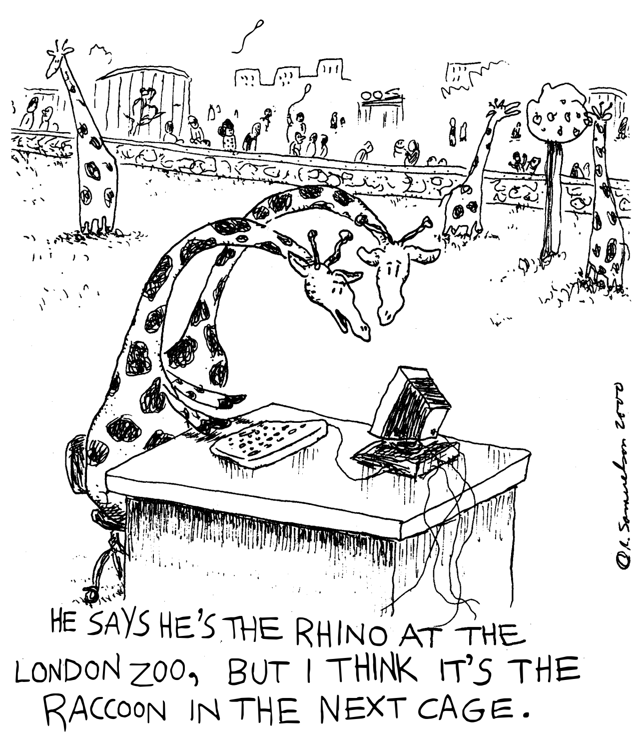
REFERENCES

1. Robert Beals, personal correspondence, 1986.
2. Arthur T. Benjamin, David J. Gaebler, and Robert P. Gaebler, A Combinatorial Approach to Hyperharmonic Numbers, preprint.
3. Louis Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Boston, 1974.
4. John H. Conway and Richard K. Guy, *The Book of Numbers*, Copernicus, 1996.
5. Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison Wesley, 1993.
6. Greg Preston, *A Combinatorial Approach to Harmonic Numbers*, Senior Thesis, Harvey Mudd College, Claremont, CA 2001.

Giraffes on the Internet

RICHARD SAMUELSON



Plotting the Escape—An Animation of Parabolic Bifurcations in the Mandelbrot Set

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An exercise assigned to a class studying fractals led to the idea of making an animation related to the Mandelbrot set. The students were to write a simple program in *BASIC* to plot orbits obtained by iterating the complex-valued function $f_c(z) = z^2 + c$ for various values of the complex parameter c. They found that the selection of c was critical; most values produced very uninteresting orbits.

In fact, at first the whole project appeared to be doomed to failure. We began with the easiest case, $c = 0$; the orbit of a point z_0 is $z_0, z_0^2, z_0^4, \ldots$. When we chose a starting point *inside* the unit circle, after one or two iterations the iterates were so close to zero that on the screen they appeared to *be* zero. When we chose a starting point *outside* the unit circle, after one or two iterations the iterates were so large in magnitude that they were off the screen. When we selected a point *on* the unit circle, after one or two iterations, because of the lack of numerical accuracy of the computer, the iterates left the unit circle and quickly headed straight for zero or infinity. The dynamics of $f_0(z) = z^2$ on the unit circle are, of course, worthy of much study and have been described extensively (see Devaney [**2**]). These difficulties did present me with a good opportunity to discuss the notions of stable and unstable orbits, and attracting and repelling fixed points.

After this first experiment, it seemed as if it might be a hopeless task to produce a complex number c where we could start with a point z_0, plot its orbit under iteration of $f_c(z) = z^2 + c$ on the computer and see an interesting and illustrative result. But it is possible! For example, with $c = -.379 + .587i$ we see an attracting fixed point and the orbits of nearby points approaching the fixed point in a nice spiral arrangement as in FIGURE 1. The program that draws pictures like this one creates larger disks early in the iteration and smaller ones later.

For $c = -.742 + .1166i$ we can see an attracting period 27 cycle. For $c = -.3905 + .5868i$ we can see a Siegel disk, which is a region in the complex plane inside of which orbits are deformed circles, invariant under iteration of f_c.

On my website (see http://phoenix.liu.edu/~aburns/orbits/colororb.html), there is a Java applet allowing the user to click on a point and see its orbit for various values of c. In the applet, you may choose from pre-selected values of c, or you can enter your own. The applet allows you to select the number of colors n; then every nth point in the orbit is colored the same color, showing the nature of the orbits in a neighborhood of an attracting period n cycle. (If all you want is to see the specific examples in this article, follow the links from the MAGAZINE's website: www.maa.org/pubs/mathmag.html.)

When I would give students an interesting value of c, they enjoyed using the applet to experiment by making very small changes in the real and imaginary parts of c. They were amazed that the smallest changes in the parameter c produced drastic, qualitative changes in the orbits. For example, try a small change in the c that produces a Siegel disk. The next question was, "How do we pick interesting values of c for ourselves?" The answer, of course, is to be found in a study of the Mandelbrot set, and that is the subject of this paper.

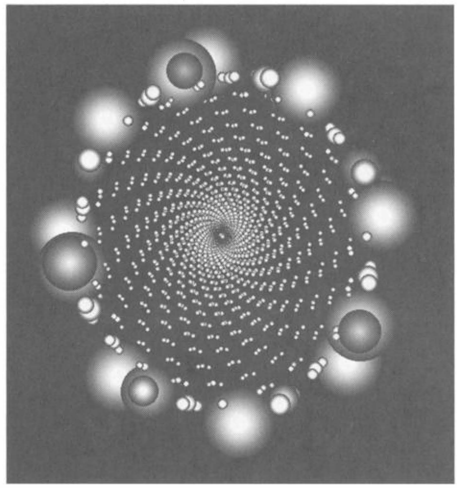

Figure 1 Orbits near an attracting fixed point

The Mandelbrot set, shown in FIGURE 2, is the set of c-values for which the orbit of 0 is bounded. In the picture, the origin is somewhere near the middle of the central cardioid, and the set is symmetrical about the real axis, which is shown horizontally, as usual. The bulbs correspond to bounded orbits of various periods.

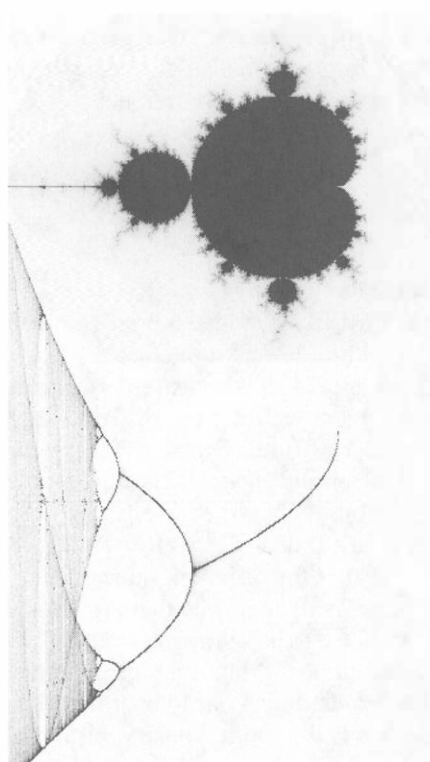

Figure 2 Shown below the Mandelbrot set is the familiar orbit diagram with the succession of period-doubling bifurcations that occur as the parameter c is decreased from $1/4$ to -2 along the real axis

Earlier in the course we had studied the dynamics of the real-valued function $f_c(x) = x^2 + c$ where both the argument x and the parameter c are real numbers. For background, see Devaney [**2**] or Ross [**6**]. The well-known orbit diagram is shown in FIGURE 2 below the Mandelbrot set. This shows, for each value of c on the horizontal axis, the values of x in the orbit of 0 after many iterations. The horizontal scales of the two diagrams are the same, for easy comparison.

If c is chosen from one of the bulbs in the Mandelbrot set along the negative real number line, then directly below it we can read from the orbit diagram the period of the attracting cycle for f_c. In the orbit diagram we can see the sequence of period-doubling bifurcations leading to chaos. These bifurcations take place as the real-valued c passes through the sequence of discs along the negative real axis and then escapes the Mandelbrot set. This suggested that we try to find other escape routes from the Mandelbrot set along which we might find a different pattern of interesting dynamics.

One difficulty with following new escape routes is that as c escapes the Mandelbrot set along paths other than the real line, we require two parameters, the real and imaginary parts of c. A second difficulty is that the attracting periodic points are not real in general, and thus it is no longer possible to plot the attractors versus c in two dimensions. A solution was to make an animation where we view the changes in dynamics over time. Finally, we put the animations together into a larger animation where we can see the variety of explosions in the fixed-point structure of f_c as c winds around the boundary of the large cardioid. Though my original intent was to examine the bifurcations that took place at parabolic points (to be defined later), the animations revealed that some of the more spectacular explosions occur as c escapes the Mandelbrot set *near* parabolic points. An animation can be seen on the website at http://phoenix.liu.edu/~aburns/orbits/nfurc800.htm.

As a bonus, the investigation of escape routes suggests a way to construct a recursive fractal resembling the Mandelbrot set that might occur if, as some of our students believe, all the numbers between 0 and 1 were rational.

Preliminaries

The family of complex functions $\{f_c(z)\}$ where $f_c(z) = z^2 + c$ and c is a complex parameter has been studied extensively. We give a brief summary of only the facts necessary to understand the algorithm. In this limited space it is not possible to cover all details and proofs, as the Mandelbrot set is an extremely complicated object. For much more about this fascinating subject we refer the reader to the excellent articles in the References, especially the paper of Bodil Branner [**1**]. If you need more background about the Mandelbrot set, see Devaney's book [**2**] or the article by John Ewing [**3**].

We are interested in the orbits $\{f_c^n(z_0)\}$ where, for a function f, f^n denotes the n-fold composition of f with itself, that is, $f^n(z) = f(f(\dots(f(z))))$, and z_0 is some initial point. It is well known that for different values of the parameter c we obtain many different types of orbits. There are orbits that are attracted to a single attracting fixed point, orbits that are attracted to an attracting cycle of period n for any integer n, orbits that travel on invariant circles, orbits that are repelled from a repelling fixed point, and chaotic orbits. When we finish our tour of the Mandelbrot set, we will see that it is easy to find a value c and a point whose orbit does just about anything we want it to do.

First we present some definitions. A point w is a *fixed point* of a function f if $f(w) = w$. If w is a fixed point for f, w is *attracting* if $|f'(w)| < 1$, *super-attracting* if $f'(w) = 0$, *neutral* if $|f'(w)| = 1$, or *repelling* if $|f'(w)| > 1$. A *cycle of period n* *for* f is an orbit $\{w_1, w_2, \ldots, w_n\}$ such that $w_{k+1} = f(w_k)$ for $k = 1, 2, \ldots, n-1$, and $w_1 = f(w_n)$. If $w_1 \neq f(w_j)$ for $j < n$ we say that this cycle has *prime period* n. (This vocabulary is common, and does not mean that n has to be a prime number.) Since a cycle of period n is obviously a cycle of period kn for any integer k, for the sake of brevity when we say period n we will mean prime period n unless otherwise noted. Note that each w_k is a fixed point of f^n . Also note that for any w_j in the cycle,

$$(f^n)'(w_j) = \prod_{k=1}^{n} f'(w_k)$$

by the chain rule. This quantity is called the *multiplier* of the cycle. The cycle is called *attracting*, *super-attracting*, *neutral*, or *repelling*, according to whether the absolute value of the multiplier γ is less than 1, equal to 0, equal to 1, or greater than 1. A neutral fixed point or cycle is called *parabolic* if the derivative or multiplier at that point or cycle is $e^{i2\pi\alpha}$ with α rational. In this paper, we concern ourselves with the bifurcations that take place at the parabolic fixed points and cycles.

DEFINITION. The *Mandelbrot set* is defined by

$$M = \{c \mid \{f_c^n(0)\} \text{ is bounded for } n \in \mathbb{N}\}.$$

Since students find this a hard concept to grasp, we emphasize that the Mandelbrot set M lies in the c plane, that is, the parameter plane. We are going to be interested in the open connected components W of M such that for all $c \in W$, f_c has an attracting fixed point or an attracting cycle of period n for some $n \in \mathbb{N}$. Such a component is called a *hyperbolic* component. We say that a fixed point w of a function f attracts a point z if

$$f^n(z) \to w \text{ as } n \to \infty.$$

A cycle $\{w_1, w_2, \ldots, w_q\}$ *attracts* a point z if $f^{qn}(z) \to w_k$ as $n \to \infty$ for one of the w_k in the cycle. The set of points attracted to an attracting fixed point or cycle is called the *basin of attraction* of the fixed point or cycle. It is easy to see that it is an open set containing the point or cycle.

Fatou proved in 1905 that for a rational function, the basin of attraction of an attracting fixed point or cycle is either the whole of $\mathbb{C}$ or it contains a critical point. A proof can be found in Devaney [**2**]. Since 0 is the only critical point of f_c, for any value of c there can be only one attracting fixed point or cycle; if an attracting fixed point or cycle exists we can find it by looking at the orbit of 0.

Attracting fixed points Let $M_1 = \{c \mid f_c \text{ has a single attracting fixed point}\}$. If $c \in M_1$, then there exists $w \in \mathbb{C}$ such that $f_c(w) = w$ and $|f_c'(w)| < 1$. This means that $w^2 + c = w$ and $|2w| < 1$ or

$$c = w - w^2 \quad \text{and} \quad |w| < \frac{1}{2}. \tag{1}$$

Solving $w^2 - w + c = 0$ gives $w = (1 \pm \sqrt{1-4c})/2$, where $\sqrt{\ }$ denotes the principal value of the square root. For $c \in M_1$ there are two fixed points:

$$w = \frac{1 - \sqrt{1-4c}}{2}$$

is the single attracting fixed point, and

$$w = \frac{1 + \sqrt{1 - 4c}}{2}$$

is a repelling fixed point. Formula (1) maps the interior of the disk $|w| < 1/2$ into the interior of M_1 and the map continues to the boundary, where $|w| = 1/2$ and $c \in \partial M_1$.

From (1) we can write

$$w = \frac{re^{i\theta}}{2}, \qquad 0 \le r < 1 \quad \text{and} \quad 0 \le \theta < 2\pi, \quad \text{and} \tag{2}$$

$$c = \frac{re^{i\theta}}{2} - \frac{r^2 e^{i2\theta}}{4}, \qquad 0 \le r < 1 \quad \text{and} \quad 0 \le \theta < 2\pi. \tag{3}$$

To get the boundary of the region (3), let $r = 1$. This gives the large cardioid we see in the pictures of M. Let $\alpha = \theta/2\pi$. Then $\theta = 2\pi\alpha$, and for $0 \le \theta < 2\pi$ we have $0 \le \alpha < 1$.

As α traverses the unit interval from 0 to 1, c winds once around the boundary of the cardioid, ∂M_1, and the corresponding neutral fixed point w winds once around the circle of radius $1/2$ centered at the origin. Notice that when $\alpha = 1/2$, $\theta = \pi$, $c = -3/4$ and $w = -1/2$. This is precisely the point at which the large disk of radius $1/4$ centered at -1 is attached to M_1 and where the first period-doubling bifurcation takes place in the orbit diagram for the real valued function f_c. For future reference we mention that the derivative at the corresponding attracting fixed point w for $c \in M_1$ is $2w = 1 - \sqrt{1 - 4c}$. The map

$$c \mapsto 1 - \sqrt{1 - 4c} \tag{4}$$

is a conformal isomorphism of M_1 onto the unit disk **D** that extends continuously to the boundary. This important map will be generalized later.

Period 2 points To find a period 2 cycle we must solve $f_c^2(w) = w$, or $(w^2 + c)^2 + c - w = 0$. Using some elementary algebra and the fact that the two fixed points already discussed are also period 2 points, we find that for any c the period 2 points are solutions of $(w^2 - w + c)(w^2 + w + c + 1) = 0$. The solutions of $w^2 - w + c = 0$ are the two fixed points already discussed. The period 2 cycle consists of the roots of the other quadratic factor, $\{w_1, w_2\}$, where $w_1 = (-1 + \sqrt{1 - 4(c + 1)})/2$ and $w_2 = (-1 - \sqrt{1 - 4(c + 1)})/2$.

Express c as in equation (3); when $r = 0$, then $c = 0$, and w_1 and w_2 lie on the unit circle, as do all the cycles of $f_0(z) = z^2$. For $\alpha = 1/2$, as r increases from 0 to 1, the attracting fixed point w and the period 2 points w_1 and w_2 all coalesce at the point $-1/2$. FIGURE 3 illustrates the paths taken by w, w_1, and w_2.

It is easy to check that for $\alpha = 1/2$ and $r < 1$, the fixed point is attracting and the 2-cycle is repelling; for $r > 1$ the fixed point is repelling and the cycle is attracting. At $r = 1$, $c = -3/4$, we have $f_c'(w) = -1$ and the multiplier is $\gamma_c = f_c'(w_1) f_c'(w_2) = 4(c + 1) = 1$. The large disk attached to M_1 at this point is a hyperbolic component of M, which we will call $M_{1/2}$; for $c \in M_{1/2}$, f_c has an attracting cycle of period 2. Just as the map (4) mapped M_1 onto the unit disk **D**, it is easy to check that the multiplier, $\gamma_c = 4(c + 1)$, gives a conformal isomorphism of $M_{1/2}$ onto **D** which extends continuously to the boundary.

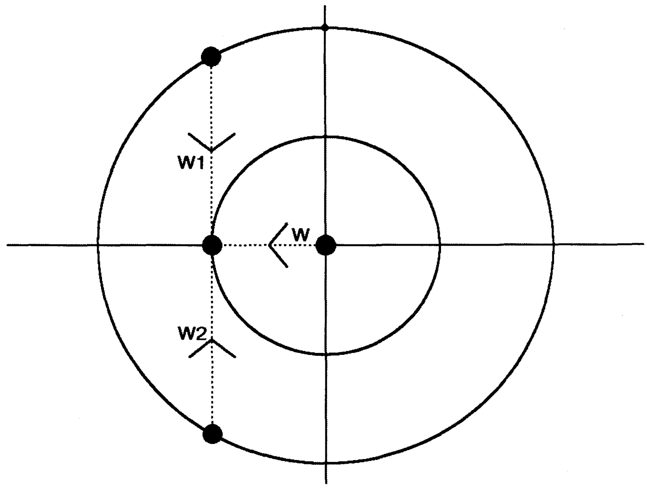

Figure 3 Paths taken by w_1, w_2, and w for $\alpha = 1/2$, $0 \le r < 1$

Period q points: the general case Again, let $c = (re^{i2\pi\alpha})/2 - (r^2e^{i4\pi\alpha})/4$, with $0 \le \alpha < 1$. Now let $\alpha = p/q$, where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$ and $p < q$. Just as in the case $\alpha = 1/2$, when $r = 1$, the point c is the point on ∂M_1 where a hyperbolic component is attached to M_1; we shall call this component $M_{p/q}$. The corresponding fixed point $w = (e^{i2\pi\alpha})/2$ is neutral. It is a parabolic fixed point. For $r < 1$, $w = (re^{i2\pi\alpha})/2$ is attracting, and there is a repelling q-cycle, $\{w_1, w_2, \dots, w_q\}$ where each w_k is a solution to $f_c^q(w) - w = 0$. Just as in the case $\alpha = 1/2$, as r approaches 1 from below, the attracting fixed point and the points in the repelling q-cycle all coalesce at the point $w = w_k = (e^{i2\pi\alpha})/2$. At this point $f_c'(w) = e^{i2\pi p/q}$ and the multiplier is

$$\gamma_c = \prod_{k=1}^{q} f_c'(w_k) = \prod_{k=1}^{q} e^{i2\pi\frac{p}{q}} = 1.$$

As r increases beyond 1, c moves into $M_{p/q}$, the fixed point w becomes repelling, and the cycle $\{w_1, w_2, \dots, w_q\}$ becomes attracting. For $c \in M_{p/q}$, f_c has an attracting cycle of period q. And once again, as in the case $p/q = 1/2$, for $c \in M_{p/q}$, the multiplier γ_c is a conformal isomorphism of $M_{p/q}$ onto the unit disk $\mathbf{D}$ which extends continuously to the boundary.

Other hyperbolic components A glance at a picture of the Mandelbrot set shows that smaller bulbs emanate from each bulb $M_{p/q}$. For c in the interior of these bulbs, f_c has an attracting cycle of some finite period. There are two types of hyperbolic components of M; they can be described intuitively as (i) primitive hyperbolic components, which look like M_1 itself (having a cusp) and (ii) those that are tangent to another hyperbolic component and whose boundary is smooth. In FIGURE 1, the primitive hyperbolic components are almost too small to be seen; one appears as a speck far to the left of the figure. For a good mathematical description, see Milnor [**4**, p. 230]. We will be interested only in the second type.

Let W be any hyperbolic component of M; then for $c \in W$, f_c has an attracting cycle of period q. W has a well-defined center, a value of c for which c is on the orbit of 0; that is, the cycle $\{0, c, (c^2 + c), \dots\}$ is super-attracting: the multiplier

$$\gamma_c = \prod_{k=1}^{q} f_c'(w_k) = 0,$$

since $f_c'(0) = 0$. A theorem of Douady, Hubbard, and Sullivan tells us that γ_c is a conformal isomorphism of W onto the open unit disk and the map extends continuously to the boundary on which $|\gamma_c| = 1$. Thus on ∂W we have

$$\gamma_c = e^{i2\pi\beta}, \qquad 0 \le \beta < 1.$$

The variable β is called the *internal argument* of W. At the point c where $\gamma_c = e^{i2\pi\beta}$, (where $\beta = s/t$, $s, t \in \mathbb{N}$, and s, t relatively prime), there is another smaller hyperbolic component V attached to W. For $c \in V$, f_c has an attracting cycle of period qt. See Peitgen and Richter [**5**]. When $\beta = 0$, $\gamma_c = 1$ and for a hyperbolic component of the second (nonprimitive) type the corresponding value of c is the point where W is attached to a larger bulb, or, in the case where W is a component attached to the boundary of the main cardioid M_1, c is the point of attachment; we have already seen that at that point we had $\gamma_c = 1$. The curves of internal argument β are those curves that are mapped by γ_c onto the rays $re^{i2\pi\beta}$, $0 \le r < 1$. For M_1 the curves of internal argument β are exactly the curves $c = (re^{i2\pi\alpha})/2 - (r^2 e^{i4\pi\alpha})/4$, $0 \le r < 1$ illustrated in FIGURE 4.

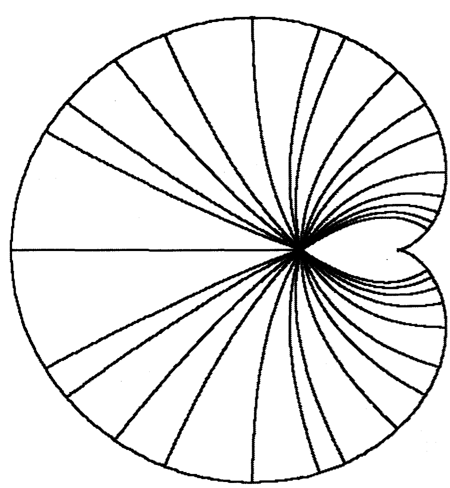

Figure 4 Escape routes from M_1: curves of internal argument p/q, $2 \le q \le 10$, $1 \le p < q$, $\gcd(p, q) = 1$

This progression of bulbs attached to bulbs suggests using recursion to construct a geometric fractal that resembles the Mandelbrot set without the hairs, filaments, and primitive hyperbolic components. It turns out to be surprisingly easy. The only problem was what radius to make each of the bulbs. Milnor [**4**, p. 288] gave the answer, but without a reason: The approximate radius of each $M_{p/q}$ is $(\sin(\pi p/q))/q^2$. We see that this is exact for $p/q = 1/2$. Compare FIGURE 5, a picture of what we will call the *fake Mandelbrot set*, with the real thing in FIGURE 1, which was generated by the pixel method described in Ewing [**3**]. In the appendix we present an algorithm for creating this fractal.

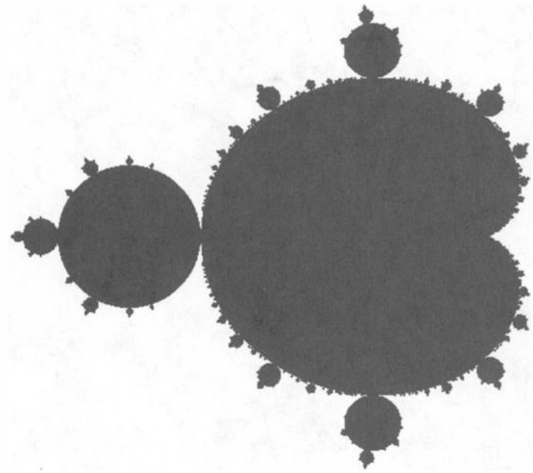

Figure 5 Fake Mandelbrot set

Escape routes

To get the familiar bifurcation diagram in FIGURE 2, c starts at the point $1/4$, then travels along the path of internal argument 0 of M_1 to the center of M_1; c then continues along the curve of internal argument $1/2$ until it reaches the point $-3/4$ where it enters $M_{1/2}$ and is now on the path of internal argument 0 with respect to $M_{1/2}$. When c reaches -1, the center of $M_{1/2}$, it jumps to the path of internal angle $1/2$. More generally, c enters a period 2^n bulb on a curve of internal argument 0, follows this curve to the center of the bulb, and leaves on a curve of internal argument $1/2$ and enters a bulb of period 2^{n+1}. Thus, while c appears to be merely travelling along a straight line, it is doing something more complicated, in terms of the internal argument.

To imitate this process, c should start at $1/4$, follow the real number line to 0, turn onto the path of internal argument p/q for M_1, $c = (re^{i2\pi p/q})/2 - (r^2e^{i4\pi p/q})/4$. When $r = 1$, c enters the bulb $M_{p/q}$. The point c should then follow the path of internal argument 0 (with respect to $M_{p/q}$) to the center of $M_{p/q}$, then turn onto the path of internal argument p/q, follow it to the budding point of a period q^2 bulb, and continue the process going through bulbs of period $q, q^2, q^3, \ldots$ before the onset of chaos. FIGURE 6 shows the approximate path for $p/q = 1/3$.

To visualize the orbit diagram for $p/q = 1/3$ we show the attractors as frames of an animation. We view the orbit of 0 as c follows the path in FIGURE 6. FIGURE 7 is a snapshot of the screen when the animation has finished. To see the true picture it is necessary to watch the animation, as points in later orbits cover points in earlier orbits. For $q > 2$ the escape routes are more complicated and finding a formula that generalizes the path taken when $p/q = 1/2$ is a problem for further research. We will describe an easier task, an animation of the changes in the dynamics of f_c as c escapes M_1 and enters one of the bulbs.

Crossing the boundary of M_1 The first step in building the animation is to draw the orbit diagram for $\alpha = p/q$, or, actually, for any value of α. As usual, let $c = (re^{i2\pi\alpha})/2 - (r^2e^{i4\pi\alpha})/4$, where α is fixed. We want to start with c inside M_1, that is, let r start at some value $r\_min < 1$. We will let r increase to some value $r\_max > 1$ in increments dr. Since the interesting dynamics occur near the boundary of M_1, to start, let $r\_min = 0.98$, $r\_max = 1.2$, and $dr = 0.002$; these values can be adjusted later.

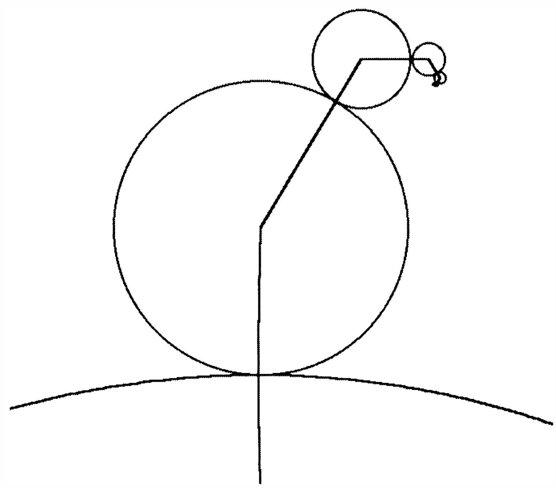

Figure 6 Escape route from M along curves of internal argument 1/3

For each value of r, plot some number n of the points $f_c^k(0)$, $k = 1, 2, \ldots, n$ in the complex plane. Figuring out the region in the plane in which to display the picture is easy if we recall that when $r < 1$, $w = (re^{i2\pi\alpha})/2$ is an attracting fixed point. At first the orbit of 0 will be attracted almost immediately to that point, and the period q-cycle will encircle that point; as r gets closer to 1 the orbit will take longer to get near that point; finally at r=1 the fixed point and cycle coalesce. As c enters a bulb and approaches the center, one of the points in the cycle moves back toward the origin. Hence the center of the screen should be placed where the fixed point and cycle coalesce, that is, $(re^{i2\pi\alpha})/2$, or $x\_center = (\cos(2\pi\alpha))/2$ and $y\_center = (\sin(2\pi\alpha))/2$; the width and height of the screen should be some number less than 2. Experiment shows that good values are 1.7, 1.4, and 0.8.

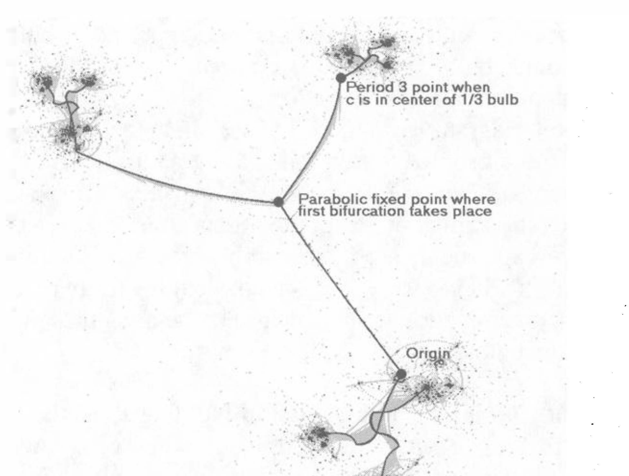

Figure 7 Orbit diagram as c follows the escape route in FIGURE 6. The orbit of 0 is plotted 400 times for each value of c. The first 350 points in the orbit are colored gray, the last 50 colored black.

We could use various color schemes to see patterns in the orbit diagram. We could color points according to the value of k. Or we could color them according to the value of r, or according to some function of k and r. For $\alpha = p/q$, using q colors and coloring the points numbered k mod q shows us clearly the order followed by the points in each orbit.

When c enters a bulb $M_{p/q}$ the initial point in the orbit is drawn almost immediately into the q-cycle. Some beautiful patterns emerge when we perturb p/q slightly; that is, let $\alpha = p/q + \delta$ for some small value of δ, for in that case c exits M briefly and then reenters. FIGURES 8, 9, 10, and 11 show the orbit diagrams for various values of α. For very small values of α the path is near the cusp of the cardioid and we see an implosion as c reenters M_1.

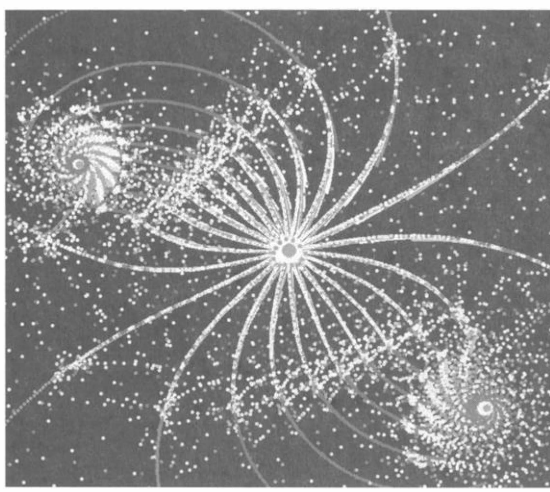

Figure 8 Here $p/q = 11/23$; first c enters $M_{11/23}$, then apparently escapes M briefly before re-entering M in the bulb $M_{1/2}$

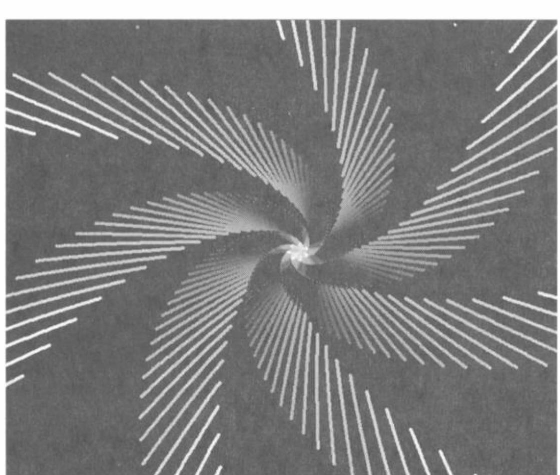

Figure 9 Close-up of $\alpha = 3/7 + .005$

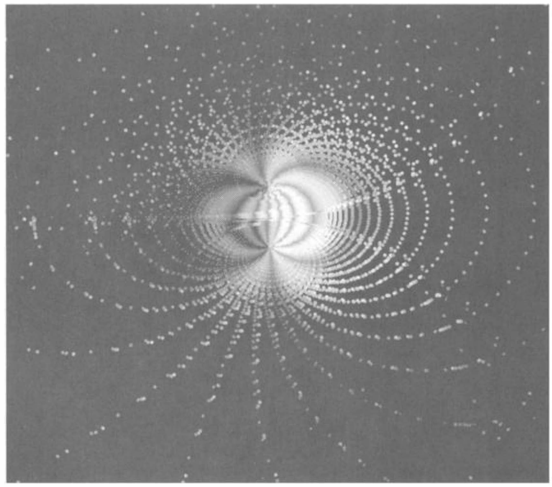

Figure 10 α very small; c escapes M near the cusp and then re-enters

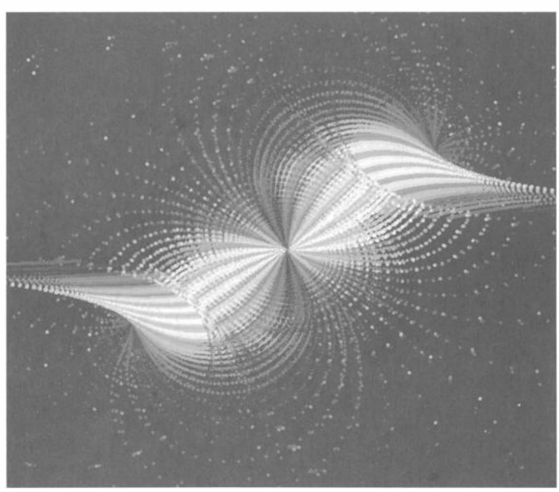

Figure 11 α near $\frac{1}{2}$

For α sufficiently irrational (see Peitgen and Richter [**5**]) it has been shown that there is a neighborhood of $w = (e^{i2\pi\alpha})/2$ called a *Siegel disk* inside of which iteration of f_c behaves like a rotation; orbits are confined to deformed circles surrounding the fixed point w that are invariant under iteration of f_c. Of course, for a computer, all numbers are rational, and even rational numbers are not all represented exactly. So, when we let $\theta = 2\pi\alpha$, the computer's π is not really π, and if we try to make α irrational we cannot really do so. However approximations work quite well; approximating π by 3.1415926 gives good results. To find a value for α that will let us see orbits near a Siegel disk we can use ratios of successive Fibonacci numbers, which we know approach $(-1 + \sqrt{5})/2$. This value of α is known to admit a Siegel disk.

Around the boundary of M_1: the animation We are ready to make the animation, an easy task now that the algorithm for the orbit diagram as a function of α is in place. Initially, my idea was to let $\alpha = p/q$, taking $q = 2, 3, 4, \ldots$ and letting p run through those integers 1 to $q-1$ that are relatively prime to q. But this method gave a very slow progression to interesting dynamics. We get a more exciting sequence of pictures by perturbing these α, letting $\alpha = 1/q + \delta$, where δ is some small number, for example 0.0015. This is the category called *almost rational* in the applet on my website.

Alternatively, we can simply let α increase from 0 to 1 in constant increments. If we choose this method, we should take care to make the increment so that the sequence of αs is not a sequence of rational numbers all having the same denominator. Another interesting progression of increments is to let α run through successive ratios of Fibonacci numbers. And, of course, the simplest way to choose values of α is to just use pseudo-random numbers between 0 and 1; this function is built-in in most computer languages.

For each α we clear the computer screen and plot the orbit diagram for that value of α. This sequence gives us the animation.

Students may enjoy thinking up new ways to animate the orbit diagrams.

Conclusion

Using the paths of internal argument α as escape routes from the Mandelbrot set allows us to visualize a series of parabolic bifurcations and to generate pictures of a variety of orbits. For example, the nature of a q^2 cycle is very different when c is chosen from a secondary bulb of internal argument p/q attached to a p/q bulb than when it is chosen from a primary p^2/q^2 bulb. In the Orbits applet, compare the orbits for $c = -.032 + .793i$ (which is in the smaller bulb of internal argument 1/3 attached to $M_{1/3}$) with orbits for $c = .34 + .073i$ (which is in $M_{1/9}$). Both values of c correspond to an attracting cycle of period 9, but the placement of points in the cycles, and hence the orbits, is very different in the two cases. In the simplest case, escaping the large cardioid by following the path $c = (re^{i2\pi\alpha})/2 - (r^2e^{i4\pi\alpha})/4$ makes it easy to find values of c near the boundary of the cardioid where exciting dynamics occur. This expression for c is also useful when teaching students how to generate interesting Julia sets. The idea behind the well-known orbit diagram for the real case suggested visualizing the orbit of 0 as the parameter changed. The idea of traveling around the boundary of M_1 came from seeing a variety of orbit diagrams and wanting my students to see them all at once.

I have always been fascinated by recursion, and studying the structure of the succession of parabolic bifurcations naturally led to my wondering how the picture would look if the only numbers between 0 and 1 were rational. This led to the construction of the fake Mandelbrot set.

Acknowledgment. I would like to give special thanks to Marc Frantz, Indiana University, Bloomington, whose critique was above and beyond that of the usual referee and whose detailed comments about escape paths were more than helpful; they enabled me to organize a collection of ideas into a logical exposition.

Appendix: Algorithm for the fractal Fake Mandelbrot set To create the fractal, we begin by drawing the boundary of M_1, $c = (re^{i\theta})/2 - (r^2e^{i2\theta})/4$ for $0 \le \theta < 2\pi$, and filling in the interior.

Then we let q start at 2, increment q by 1 until $1/q^2$ is less than one pixel width. For each q, for each p that is relatively prime to q we are going to draw a disk of radius $(\sin(\pi p/q))/q^2$ attached to the boundary of M_1 at the point $c = (2re^{i2\alpha} - r^2e^{i4\alpha})/4$. To draw each of these disks we use a recursive function whose arguments are: *radius*, *internal_angle*, and *center*. The stopping point for the recursion is when *radius* is less than one pixel width. The function first draws the disk and then calls itself with the new arguments:

$$new\_radius = radius * \frac{\sin(\pi p/q)}{q^2}$$
$$new\_angle = angle + \pi + 2\pi p/q$$
$$new\_center = center + (radius + new\_radius) * e^{i*new\_angle}$$

An amusing variation is to color each of the bulbs a color based on the period of the attracting cycle of f_c for c in that bulb.

REFERENCES

1. Bodil Branner, The Mandelbrot set, in R. Devaney and Linda Keen, eds., *Chaos and Fractals* (Proceedings of symposia in Applied Mathematics, vol. 39), Amer. Math. Soc., Providence, RI, 1988, 75–105.
2. Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1987.
3. John Ewing, Can we see the Mandelbrot set?, *College Math. J.* **26**:2 (1995), 90–99.
4. J. Milnor, Self-similarity and hairiness in the Mandelbrot set, in M. Tangora, ed., *Computers in Geometry and Topology* (Lecture Notes in Pure and Applied Mathematics, vol. 114), Dekker, New York, 1989, 211–257.
5. H.-O. Peitgen and P.H. Richter, *The Beauty of Fractals*, Springer-Verlag, 1986, 56–62.
6. Chip Ross and Jody Sorensen, Will the real bifurcation diagram please stand up!, *College Math. J.* **31**:1 (2000), 2–14.

Playing with Fire

RICHARD SAMUELSON

NOW, IF WE COULD ONLY *HARNESS* THAT ENERGY IN SMALL AMOUNTS AT ALTERNATING INTERVALS INSIDE A METAL CYLINDER WE'D HAVE AN INTERNAL COMBUSTION ENGINE.

NOTES

Four-Person Envy-Free Chore Division

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In this article we explore the problem of *chore division*, which is closely related to a classical question, due to Steinhaus [**10**], of how to cut a cake fairly. We focus on *constructive* solutions, that is, those obtained via a well-defined procedure or algorithm. Among the many notions of fairness is *envy-freeness*: an envy-free cake division is a set of cuts and an allocation of the pieces that gives each person what she feels is the *largest* piece. It is non-trivial to find such a division, since the cake may not be homogeneous and player valuations on subsets of cake will differ, in general. Much progress has been made on finding constructive algorithms for achieving envy-free cake divisions; most recently, Brams and Taylor [**3**] produced the first general n-person procedure. The recent books by Brams and Taylor [**4**] and Robertson and Webb [**8**] give surveys on the cake-cutting literature.

In contrast to cakes, which are desirable, the dual problem of chore division is concerned with dividing an object deemed *undesirable*. Here, each player would like to receive what he considers to be the *smallest* piece of, say, a set of chores. This problem appears to have been first introduced by Martin Gardner [**6**].

Much less work has been done to develop algorithms for chore division than for cake-cutting. Of course, for 2 people, the familiar *I-cut-you-choose* cake-cutting procedure also works for dividing chores: one cuts the chores and the other chooses what she feels is the smallest piece. Oskui [**8**, p. 73] gave the first envy-free solutions for chore division among 3 people. Su [**12**] developed an envy-free chore-division algorithm for an arbitrary number of players; however, it does not yield an exact solution, but only an ϵ-approximate one. There appear to be no exact envy-free chore-division algorithms for more than three players in the literature; in unpublished manuscripts, Brams and Taylor [**2**] and Peterson and Su [**7**] offer n-person algorithms but these are not bounded in the number of steps they require.

In this article, we develop a simple and bounded procedure for envy-free chore division among 4 players. The reader will find that many of the ideas involved—moving knives, trimming and lumping, and a notion of "irrevocable advantage"—provide a nice introduction to similar techniques that arise in the literature on *fair division* problems. As a warm-up to some of these ideas, we also present a 3-person solution that is simpler and more symmetrical than the procedure of Oskui.

We assume throughout this paper that chores are infinitely divisible. This is not unreasonable, as a finite set of chores can be partitioned by dividing up each chore (for instance, a lawn to be mowed could be divided just as if it were a cake), or dividing the time spent on them. We also assume that player valuations over subsets of the chores are additive, that is, no value is destroyed or created by cutting or lumping pieces together. (The proper context for modeling player valuations is measure theory, but we can avoid that for the purposes of this article.)

A 3-person chore-division procedure We first describe a simpler 3-person chore-division procedure, which introduces some ideas that are important later.

Our 3-person chore-division procedure relies on Austin's procedure **[1]** for dividing a cake into two pieces so that each of two players believes it is a 50-50 split. For completeness, we review it here. Let one player hold two knives over the cake, with one at the left edge, such that the portion of cake between them is what she believes to be *exactly* half. If the second player agrees that it is exactly half, we are done. Otherwise, let the first player move the knives across the cake from left to right, keeping the portion between them exactly half (in her estimation), until the second player agrees it is exactly half. (There must be such a point because when the rightmost knife reaches the right edge, the leftmost knife must be where the rightmost knife began, hence the second player must by that point have changed preferences.) At this point cuts are made and the pieces of cake outside the knives are lumped together, yielding two pieces that both players agree are exactly equal.

Austin's procedure is an example of what is sometimes called a "moving-knife" procedure in the cake-cutting literature **[5]**. Our 3-person chore-division algorithm is also a moving-knife procedure. The key idea is to divide the chores into *six* pieces and assign each player two of the pieces that he feels are at least as small as each pair of pieces the other players receive.

A THREE-PERSON ENVY-FREE CHORE-DIVISION PROCEDURE

Step 1. Divide the chores into three *portions* using any 3-person envy-free cake-division procedure (that guarantees players a piece they think is largest), such as the Stromquist moving-knife procedure **[11]**. Now label each portion by the name of the player to whom the cake-division procedure would assign that portion (this player believes that portion is *largest*).

Step 2. Let player i divide portion i into 2 pieces (which she feels is exactly half) and assign those pieces to the *other* two players such that they each feel they have received no more than half of portion i. (This can be achieved via Austin's procedure: letting player i and one other player, say j, agree on a 50-50 split, let the remaining player choose the half she thinks is smallest, and give the other half to j.)

Step 3. Repeat Step 2 for each player, then end the procedure.

We now verify that each player has been assigned *two* out of six total pieces such that each feels her share is *smallest*.

Call the players i, j, and k. Player i will not envy player j because one piece of each of their pairs came from the portion labelled k, and i feels her half of that portion was no larger than j's. As for her other piece, player i feels it was no more than half of the portion it came from, and therefore cannot be as large as player j's other piece, which i felt was exactly half of the *largest* portion. The same argument holds for any permutation of i, j, and k. See FIGURE 1.

This procedure requires at most 8 cuts (Step 1 uses 2 cuts, and Austin's procedure uses at most 2 cuts each time it is applied; in FIGURE 1, some pieces may have been reassembled for simplicity). It is also less complicated than the discrete procedure of Oskui **[8]**. There are some 3-person moving-knife schemes that require fewer cuts **[8, 9]**, but our approach is distinguished by being symmetric with respect to the players and being based on a cake-cutting procedure. The former property simplifies

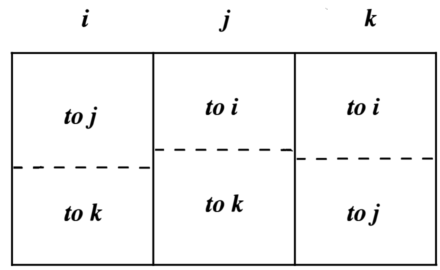

Figure 1 An envy-free assignment of six pieces (of chores) to three people

the verification of envy-freeness, while the latter property may help in generalizing the scheme to more players via known cake-cutting algorithms.

A 4-Person Chore-division Procedure We now describe our 4-person chore-division procedure, which is also a moving-knife procedure and requires at most 16 cuts. It draws ideas from both the Brams-Taylor-Zwicker 4-person envy-free moving-knife scheme for cakes [**5**] and the Oskui 3-person envy-free discrete chore-division scheme [**8**]. We also show how the notion of *irrevocable advantage*, important in cake-cutting [**4**], can be applied in chore division.

Suppose the players are named Alice, Betty, Carl, and Debbie. For convenience, we assume the chores are a rectangular block that may be divided by vertical cuts. Let Alice and Betty divide the chores into four pieces they both agree are all equal, by performing three applications of Austin's procedure (using at most 6 cuts).

Call the pieces X_1, X_2, X_3, and X_4. Note that if Debbie and Carl disagree on which piece is the smallest, we can immediately allocate the pieces. Thus we may assume they agree that one piece is *strictly* smaller than the others, say X_4. Then each person thinks the following:

$$\begin{aligned} \text{Alice:} &\quad X_1 = X_2 = X_3 = X_4 \\ \text{Betty:} &\quad X_1 = X_2 = X_3 = X_4 \\ \text{Carl:} &\quad X_4 < X_1, X_2, X_3 \\ \text{Debbie:} &\quad X_4 < X_1, X_2, X_3. \end{aligned}$$

Now, for each of X_1, X_2, and X_3, let Debbie and Carl mark how they would trim them to make them the same size as X_4. As each piece is rectangular, assume the trimmings are marked from the top edge, so that a person receives the piece below her mark. See FIGURE 2. Hence, we can speak of one mark as being *higher than* another. The following procedure will yield an envy-free chore division (we've already described Step 1):

A FOUR-PERSON ENVY-FREE CHORE-DIVISION PROCEDURE

Step 1. Let Alice and Betty use three applications of Austin's procedure (6 cuts) to obtain 4 pieces (X_1, X_2, X_3, X_4) that Alice and Betty believe are exactly equal in size. If Carl and Debbie disagree on which piece is smallest, then allocate the pieces accordingly and end the procedure. Otherwise, call X_4 the piece that Carl and Debbie agree is smallest.

Step 2. Let Carl and Debbie mark X_1, X_2, X_3 where they would cut them to create ties for smallest with X_4. Without loss of generality, suppose Debbie has more marks higher than Carl's. Trim the pieces at the higher marks (3 cuts), and set aside the trimmings.

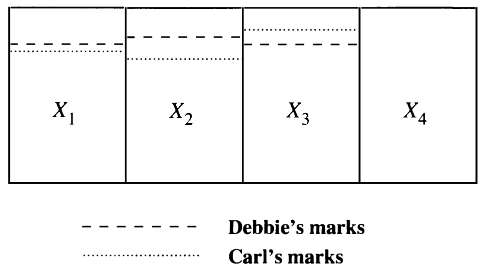

Figure 2 This figure shows a possible set of markings made in Step 1 of our 4-person procedure

Step 3. Let Betty add back to one piece some of the corresponding trimming (1 cut) to create a two-way tie for smallest piece.

Step 4. Let players choose from these pieces in the order Alice-Betty-Carl-Debbie, with Betty required to take the added-back piece if Alice didn't, and Carl required to choose a piece trimmed at his marking if it is available. This will allocate everything except the trimmings in an envy-free fashion.

Step 5. Divide the trimmings (6 cuts), exploiting an irrevocable advantage of Betty over whomever receives X_4. (The concept of an irrevocable advantage is defined and explained later.) This will allocate the trimmings in an envy-free fashion.

We now verify that this yields an envy-free solution. We assumed in Step 2 that Debbie has more higher marks than Carl (if not, just reverse the roles of Carl and Debbie in what follows). This produces two cases, the first in which Debbie has three higher marks and the second in which Debbie has two.

Case I: Debbie has three higher marks. Assume that Debbie's marks are all at or above Carl's marks. Following Step 2, let Debbie trim X_1, X_2, and X_3 at her marks to obtain a four-way tie for the smallest piece. Call the trimmed pieces X_1', X_2', and X_3', and the trimmings T_1, T_2, and T_3, which are set aside for later. At this point, each person thinks:

$$\begin{aligned}
\text{Alice:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Betty:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Carl:} &\quad X_4 \leq X_1', X_2', X_3' \\
\text{Debbie:} &\quad X_1' = X_2' = X_3' = X_4.
\end{aligned}$$

Carl must believe X_4 is the smallest, or tied for the smallest, because his marks were all at or below Debbie's (meaning he believes more should be trimmed to make them equal to X_4).

Of the remaining pieces X_1', X_2', X_3', suppose without loss of generality that Betty believes $X_3' \leq X_2' \leq X_1'$. Following Step 3, let Betty return some of the trimmings T_3 to X_3' to create a two-way tie for the smallest piece. (We still call the modified piece X_3', and its trimmings T_3.) Thus player valuations change:

$$\begin{aligned}
\text{Alice:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Betty:} &\quad X_2' = X_3' \leq X_1' < X_4 \\
\text{Carl:} &\quad X_4 \leq X_1', X_2', X_3' \\
\text{Debbie:} &\quad X_1' = X_2' = X_4 < X_3'.
\end{aligned}$$

Following Step 4, Alice chooses first (hence is envy-free), then Betty, who is required to take X_3' if it was not chosen by Alice. Since Betty has at least one of X_2', X_3' to choose from, she is envy-free. Then Carl chooses, and will clearly still have his smallest piece X_4 available. Debbie will have one of her three smallest pieces available because Betty took X_3' if Alice did not. Thus all are non-envious of the portions they have recieved so far.

Dividing the Trimmings. The trimmings still need to be divided and assigned. Without loss of generality, suppose that Betty chose X_3' in the procedure above. Then Betty thinks

$$\text{Betty:} \quad T_1, T_2 \leq T_3.$$

Note that because Betty believed $X_3 = X_4$, she could receive all of T_3 and still not envy Carl. In fact, by the above inequality she could receive $\frac{1}{3}(T_1 + T_2 + T_3) \leq T_3$ and still not envy him. We will say that Betty has an *irrevocable advantage* over Carl with respect to the trimmings.

So, lump all the trimmings together (say, $T = T_1 + T_2 + T_3$), and let Alice and Debbie use Austin's procedure to divide T into four pieces that they both agree are all equal. Then let the players choose in the order Carl, Betty, and then (in any order) Debbie and Alice.

With respect to the trimmings, Carl will envy no one because he chooses first. Betty, choosing the smallest of the remaining three pieces, will have a piece that she believes is at most $\frac{1}{3}T$ and therefore will not envy Carl. Alice and Debbie will not envy Betty or Carl because they think all four pieces are equal. Thus the trimmings can be divided in an envy-free fashion.

Case II: Debbie has two higher marks. Assume now that Debbie has two marks at or above Carl's marks. Without loss of generality suppose that Carl has a higher mark on X_3 than Debbie, as in FIGURE 2. Following Step 2, let cuts be made at all three highest marks. Then

$$\begin{aligned}
\text{Alice:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Betty:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Carl:} &\quad X_3' = X_4 \leq X_1', X_2' \\
\text{Debbie:} &\quad X_1' = X_2' = X_4 < X_3'.
\end{aligned}$$

Following Step 3, let Betty create a two-way tie for the smallest piece (as before) by returning to the smallest piece some of the corresponding trimmings. She may add either to X_1', X_2' or X_3'. The X_1' and X_2' cases are equivalent, so we have two subcases.

Suppose Betty adds to X_3' until it is as large as, say, X_2'. Then

$$\begin{aligned}
\text{Alice:} &\quad X_1', X_2', X_3' < X_4 \\
\text{Betty:} &\quad X_2' = X_3' \leq X_1' < X_4 \\
\text{Carl:} &\quad X_4 \leq X_1', X_2', X_3' \\
\text{Debbie:} &\quad X_1' = X_2' = X_4 \leq X_3'.
\end{aligned}$$

These inequalities are identical to those in Case I, and thus our procedure works in the same way. Moreover, the trimmings can be handled just as before, since Betty has an irrevocable advantage over Carl (who receives X_4).

Otherwise, suppose Betty adds to X_1' until it is as large as, say, X_2' (the X_3' case is similar). Then

$$\begin{aligned} \text{Alice:}\quad & X_1', X_2', X_3' < X_4 \\ \text{Betty:}\quad & X_1' = X_2' \leq X_3' < X_4 \\ \text{Carl:}\quad & X_3' = X_4 \leq X_1', X_2' \\ \text{Debbie:}\quad & X_2' = X_4 \leq X_1', X_3'. \end{aligned}$$

Following Step 4, let Alice choose first. When Betty chooses, she will have one of her two smallest pieces available (and will take X_1' if available as the procedure requires). Next, the procedure requires Carl to take X_3' if available (since it was the piece trimmed at his marking) and otherwise he recieves X_4; either way he believes he has the smallest piece. As X_1' and X_3' are allocated by this point, we know that Debbie will receive either X_2' or X_4, and hence is also envy-free. For the trimmings, note that Betty has an irrevocable advantage over whomever receives the X_4 piece, so the trimmings can be divided using the method discussed earlier.

This concludes the verification of envy-freeness for all cases. Note that we could alternately have presented tables for each case that list envy-free assignments for Betty, Carl, and Debbie given what Alice chose first. However, remembering those tables would not be as easy as remembering the steps of our procedure.

A bounded n-person procedure? Our procedure gives the first known bounded procedure for 4-person envy-free chore division, requiring at most 16 cuts. Actually, this can be reduced to 15 cuts with a modification much like Brams-Taylor-Zwicker's 5-cut modification **[5]** of the triple application of Austin's procedure.

Although the reader may be tempted to try to further reduce the number of cuts needed for 4-person envy-free chore division, progress in this direction is not as important as the more compelling problem of finding any bounded procedure for more than 4 players. While there do exist finite n-person envy-free chore-division procedures (**[2]**, **[7]**), these are *not bounded* in the number of steps or cuts, that is, depending on player preferences, they could take arbitrarily long to resolve. For cake-cutting as well as chore division, the existence of bounded n-person envy-free division procedures remains a major unsolved problem that will probably require new techniques.

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REFERENCES

1. A. K. Austin, Sharing a cake, *Mathematical Gazette* **6**:437 (1982), 212–215.
2. S. J. Brams and A. D. Taylor, *An envy-free cake division algorithm*, Economic Research Reports, C. V. Starr Center for Applied Economics, New York University, 1992.
3. S. J. Brams and A. D. Taylor, An envy-free cake division protocol, *Amer. Math. Monthly* **102**:1 (1995), 9–18.
4. S. J. Brams and A. D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge University Press, 1996.
5. S. J. Brams, A. D. Taylor, and W. S. Zwicker, A moving-knife solution to the four-person envy-free cake-division problem, *Proc. Amer. Math. Soc.* **125** (1997), 547–554.
6. M. Gardner, *aha! Insight*, W. F. Freeman and Co., New York, 1978.
7. E. Peterson and F. E. Su, *Exact procedures for envy-free chore division*, HMC Mathematics Department technical report, http://www.math.hmc.edu/~su/papers.html, 1998.
8. J. M. Robertson and W. A. Webb, *Cake-Cutting Algorithms: Be Fair If You Can*, A. K. Peters Ltd., 1998.
9. F. W. Simmons, personal communication, 1998.
10. H. Steinhaus, The problem of fair division, *Econometrica* **16**:1 (1948), 101–104.
11. W. Stromquist, How to cut a cake fairly, *Amer. Math. Monthly*, **87** (1980), 640–644.
12. F. E. Su, Rental harmony: Sperner's lemma in fair division, *Amer. Math. Monthly*, **106** (1999), 930–942.

The Consecutive Integer Game

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Two previously unacquainted contestants, Xeno (X) and Yvonne (Y), sit at a table facing each other and play a game. You are the referee. Once the spectators are all seated, you randomly choose two *positive, consecutive integers* and, with a black marker, write one of the numbers on Xeno's forehead and the other number on Yvonne's forehead. Each player can read the other's number but not his or her own. [For example, you give Xeno 2475 and Yvonne 2476. Xeno sees 2476 and knows that his number is either 2475 or 2477, but he doesn't know which.] The first player to figure out his or her own number wins. The game proceeds in rounds.

ROUND 1 You say, "Raise your hand if you know your number."

If no hand is raised, we go on to Round 2.

ROUND 2 You say, "Raise your hand if you know your number."

The players can have all the time they like. When they both concede that they don't know their own numbers, we go on to the next round.

ROUND 3 You say, "Raise your hand if you know your number."...

There are no mirrors, signals or other communication beyond what has been explicitly stated. The game proceeds round after round, hour after hour. The players begin to look haggard and the crowd grows restless. A small boy complains that this game is boring, but is quickly silenced when his father suggests that they go watch the cricket game instead. Then suddenly, after many rounds, Yvonne jumps up, waves her hands wildly, and shouts,

"I know! I know! My number is 2476!!"

The crowd roars with excitement as the bewildered referee hands her the coveted Consecutive Integer Game Trophy.

How did Yvonne figure out her number? More generally, if the chosen numbers are n and $n + 1$, is there always a winner? Who will it be? After how many rounds? The interested reader might well spend a few minutes pondering these questions before going further.

A preliminary analysis of the game Instead of big numbers like 2475 and 2476, suppose X has a 1 and Y has a 2. X doesn't know whether his number is 1 or 3. But Y sees a 1 and, knowing that her number is a positive integer, immediately concludes that it must be 2. Accordingly, she raises her hand and wins in Round 1. Now suppose Y has a 2 and X has a 3. Neither knows his or her own number in Round 1. But X knows that he has either a 1 or a 3, and that Y didn't win in the Round 1. So, in Round 2, he concludes that his number must be 3.

Suppose that X has a 3 and Y has a 4. Y knows that her number is either 2 or 4. Having read the previous paragraph she knows that if her number were 2, then X would have figured out his number and won in Round 2. Since Round 2 ended with X failing to do this, she immediately concludes that her number is 4 and wins in Round 3. This appears to be a nice little exercise in mathematical induction.

THEOREM 0. *If the two numbers are n and $n+1$, then the player whose number is $n+1$ will win in Round n.*

The proof is not as simple as it first appears since there are really two different facts to be proven. We must show that the player with $n+1$ will know his or her number in Round n, and we must also show that neither player can deduce their own number any sooner. We prove each of these facts by induction.

LEMMA A. *If X's number is n and there is no win before Round n, then Y's number is $n+1$.*

Proof. Since the numbers must be positive, this is clearly true for $n = 1$. Assume that it is true for $n = k$. Suppose that X's number is $k+1$, and that Round $k+1$ arrives with no previous winner. Y's number is either k or $k+2$. If it were k, then since the statement is true for k, player X would have seen a k and won in Round k. Since he didn't, Y's number must be $k+2$. ■

LEMMA B. *If Y's number is greater than n and there is no win before Round n, then X will not know his own number in Round n.*

Proof. This is clearly true for $n = 1$ since there will be two distinct possible numbers that X might have. Assume that it is true for $n = k$. Suppose that Y's number m is greater than $k+1$ and that there is no win before Round $k+1$. Since $m > k+1$, we also have $m > k$ so, by induction, X did not know his number in Round k. We must argue that X gained no new information in Round k that might have allowed him to deduce his number in Round $k+1$. The only information he received in Round k

was that Y did not at that point know her number, m. So we must argue that X would have already known, at start of Round k, that Y did not yet know her number. But this follows by induction since X's number is either $m-1$ or $m+1$, and both are greater than k. ■

Proof of Theorem 0. Assume X has n and Y has $n+1$. By Lemma A, player Y will know her number in Round n. By Lemma B, neither player will know their number before Round n and X will not even know his number in Round n. Thus Y will win in Round n. ■

We now have a relatively simple solution to what seemed initially to be an impossible problem. Or do we ...? Read on!

A careful analysis of the preliminary analysis Unfortunately all three of these purported proofs suffer from a similar flaw. We have in each case failed to distinguish between what *is true* and what *the players know to be true*. For example, let $S(n)$ be the statement of Lemma A. Induction tells us only that $S(k)$ is true, whereas the argument requires the assumption that X knows that $S(k)$ is true. Similarly, let $T(n)$ be the assertion of Lemma B. The last statement of the proof requires not only that $T(k)$ is true, but that X knows that $T(k)$ is true. Theorem 0 itself does not follow from the truth of Lemmas A and B alone; a second look at the proof shows that it requires that player X knows that Lemma A is true.

We will see that it is still possible to give a proof of Theorem 0, but that will require some additional hypotheses about the knowledge and abilities of the two players. It is tempting to add some assumption to the effect that both players are skilled logicians. Besides being vague, this would not be sufficient, as we would have to conclude that a skilled logician is necessarily able to prove both induction assumptions, which we ourselves have not yet proven. It would be sufficient to strengthen the assumptions to say that both players can prove $S(n)$ and $T(n)$ for all n. But that would be absurd, as it would mean that we are assuming in advance that both lemmas are true!

To see what might be required, suppose that you sit down to play the consecutive integer game and you see the number 4 on your opponent's forehead. Round 4 arrives with no previous win, and you would like to conclude that your number is 5. In order to do so, you must be confident that your number is not 3. This requires that you be certain that your opponent, seeing the number 3, would, in Round 3, have been able to reason as follows:

> *My number is either 2 or 4. If it were 2, then you would have known that your number was either 1 or 3. Since I didn't know my number in Round 1, you would have known that your number was not 1 and therefore must have been 3, allowing you to win in Round 2. Since you didn't win in Round 2, my number must not be 2 so it must be 4. Therefore I will raise my hand and win in Round 3.*

Clearly some players would be able to think this through while others would not. If the number you see is less than 4, then you would be more confident that your opponent could draw the necessary conclusions. If it were much more than 4, you would need to assume that your opponent was able to carry out much more complex reasoning, which itself required assumptions about *your* reasoning ability. Thus, any satisfactory definition of a *player* will require some explicit statement that we can all understand as to just what it is that a player is supposed to be able to do. In the next section, we will give such an explicit statement, and then show that Theorem 0 is true provided that the players meet our qualifications.

A careful analysis of the game In order to document a player's abilities, we might imagine establishing a **Consecutive Integer Academy**, called the **CIA** for short. Any potential players are free to enter the CIA, but they must pass appropriate examinations in order to progress. The CIA will administer a series of examinations,

$$\text{Exam } 0, \text{Exam } 1, \text{Exam } 2, \text{Exam } 3, \ldots, \text{Exam } \infty,$$

that students can take to prove their abilities. It will be run according to the following rules.

Rules of the CIA

1. Each student must wear a badge showing the number of the highest exam he or she has passed.
2. A student must pass Exam k before taking Exam $k+1$.
3. Exam ∞ may be taken by any student at any time in lieu of all of the remaining exams.

Students graduate when they pass Exam ∞, whereupon they are awarded an ∞–badge branding them as **CIA Certified Players** of the Consecutive Integer Game.

The content of the exams never changes, and is a matter of public record. Let X and Y represent two students playing the Consecutive Integer Game.

Exam 0 Demonstrate that you can count, that you know the rules of the game, and the rules of the CIA.

Exam 1 Prove these theorems:

THEOREM A_1. *If X's number is 1, then Y's number is 2.*

THEOREM B_1. *If Y's number is greater than 1, then X will not know his own number in Round 1.*

Exam 2 Prove these theorems:

THEOREM A_2. *Assume that X has passed Exam 1. If X's number is 2 and there was no win in Round 1, then Y's number is 3.*

THEOREM B_2. *Assume that X has passed Exam 1. If Y's number is greater than 2 and there is no win in Round 1, then X will not know his own number in Round 2.*

Exam n, where $n \geq 3$. Prove these theorems:

THEOREM A_n. *Assume that X has passed Exam* $n-1$ *and that Y has passed Exam* $n-2$. *If X's number is n and there was no win before Round n, then Y's number is* $n+1$.

THEOREM B_n. *Assume that X has passed Exam* $n-1$ *and that Y has passed Exam* $n-2$. *If Y's number is greater than n and there is no win before Round n, then X will not know his own number in Round n.*

Exam ∞ PART A. Prove that Theorem A_n is true for every positive integer n.

PART B. Prove that Theorem B_n is true for every positive integer n.

We will now prove Theorems A_n and B_n for an arbitrary positive integer n, as is required to pass Exam ∞. Doing so will, in particular, demonstrate that it is indeed possible to graduate certified players. Of course, we do ask that the reader keep these proofs confidential.

Theorem sequences A_n and B_n are each defined in three cases, so we need to give six different arguments. While these arguments will exhibit a rather repetitive structure, they will also grow in subtlety and complexity. Our hope is that mastery of the earlier ones will help the reader follow the later ones. Ultimately, we will prove Theorem C, which is exactly our failed Theorem 0 with a hypothesis added to make it true!

THEOREM A. *For every positive integer n, Theorem A_n is true.*

Proof. Theorem A_1 is true since the numbers are positive and consecutive.

To prove Theorem A_2, assume that X's number is 2, that X has passed Exam 1 and that there was no win in Round 1. Then Y's number is either 1 or 3. If it were 1, then X—having passed Exam 1—would have known that his number was 2 in Round 1. Since there was no win in Round 1, he didn't know his number. Therefore Y's number must be 3.

Now let $n \geq 3$. To prove Theorem A_n, assume that X has passed Exam $n-1$, that Y has passed Exam $n-2$, that X's number is n and that there was no win before Round n. Then Y's number is either $n-1$ or $n+1$. Suppose Y's number were $n-1$ and consider what X knew in Round $n-1$. By Rules 1 and 2, Y wears a badge indicating that she has passed Exam $n-2$. By Rule 2, X has also passed Exam $n-3$, and there was no win before Round $n-1$. Since X has passed Exam $n-1$ by proving Theorem A_{n-1}, he would have known that his number was n in Round $n-1$. This was not the case, since there was no win before Round n. Therefore Y's number could not be $n-1$; it must be $n+1$. ■

Notice that Theorem A_1 is very easy to prove and Theorem A_2 is only slightly more involved. The proof of Theorem A_3 is essentially the same as the proof of Theorem A_n for each $n \geq 3$. The same is true of the proofs for the sequence Theorems B_n, which are somewhat more involved than their Theorem A_n counterparts.

THEOREM B. *For every positive integer n, Theorem B_n is true.*

Proof. Theorem B_1 is true since, in Round 1, the only information available to X is that his number differs from Y's number by 1.

To prove Theorem B_2, assume that X has passed Exam 1, that Y's number is $m > 2$, and that there is no win in Round 1. Then X did not know his number in Round 1. The only new information that X gained in Round 1 that might have helped him determine his own number in Round 2 is that Y did not know her number in Round 1. However X already knew this fact at the beginning of Round 1. Indeed, X knew that his number was at least $m-1 > 1$ and he had passed Exam 1 by proving Theorem B_1, telling him that Y would not know her number in Round 1. Thus X still did not know his number in Round 2.

Now let $n \geq 3$. To prove Theorem B_n, assume that there was no win before Round n and that Y's number is $m > n$. Thus X did not know his own number in Round $n-1$. We must argue that X gained no new information in Round $n-1$ that might allow him to deduce his number in Round n. The only information he received in Round $n-1$ was that Y did not at that point know her number, m. So we must argue that X already knew, at the start of Round $n-1$, that Y did not yet know her number.

Since X has passed Exam $n-1$, he has proven Theorem B_{n-1}. To show that X knew in advance that Y would not know her number in Round $n-1$, we verify the hypotheses of Theorem B_{n-1} (with the names X and Y reversed). Since Y wears a badge certifying that she has passed Exam $n-2$, player X knows that she has proven Theorem B_{n-2}. By Rule 2, player X has passed Exam $n-3$. Since Y's number is m, the smallest number X could have is $m-1$. Since $m > n$, we have $m-1 > n-1$.

Applying Theorem B_{n-1}, player X could therefore conclude at the start of the game that Y would not know her number in Round $n - 1$. Thus X still did not know his number in Round n. ■

We can now prove an appropriately amended version of Theorem 0.

THEOREM C. *Assume that X and Y each wear an ∞–badge, indicating that they are CIA Certified Players. If they play the Consecutive Integer Game with numbers n and $n + 1$, then the player with $n + 1$ will win in Round n.*

Proof. Suppose X has n and Y has $n + 1$. By Theorem B there will be no winner before Round n, and X will not know his number in Round n. Since Y has passed Exam 0, she is able to count to n. Since she has passed Exam n, she has proven Theorem A_n. Looking at X's badge, Rule 3 tells her that X has passed Exam $n - 1$. Applying Theorem A_n, she can conclude in Round n that her number is $n + 1$. Consequently she will win in Round n. ■

The fifty guilty wives The Consecutive Integer Game arose as a variant of a problem from the folklore of mathematics. I heard it some years ago from a colleague who presented it as an amusing application of mathematical induction, but I have no knowledge of its original source. It is based on a story about an appallingly sexist place and time, where marital infidelity was commonly viewed as a woman's sin and a man's triumph. When it occurred, the guilty wife would carefully hide her shame but the guilty husband would boast to all the other men in the village. Well, not quite *all* the other men. They certainly didn't tell the High Priest. And because these men were not very courageous, they would conspire to keep the transgression a secret from the guilty wife's own husband. The result was that each husband knew exactly which wives were guilty and which were not, with the one exception of his own wife. Neither threats nor bribes would elicit information about his own wife from any other husband.

It was the duty of the High Priest to see that marital fidelity was observed, and to take appropriate action when it was violated. But this High Priest found himself left with the following perplexing situation. For each husband X and wife Y,

- if X was married to Y, then he was willing to participate in her conviction and punishment but he had no way to know if she was guilty;
- if X was not married to Y, then he was unwilling to participate in her conviction or punishment although he did know if she was guilty.

The High Priest needed to find some way to trick the husbands into divulging what they knew to the other husbands who would then administer justice. Fortunately he was a logician as well as a priest. Though not in the gossip loop, he did have spies, and as a result he knew that some marital infidelities had occurred in the village. But he didn't know exactly who was involved. So he put together a diabolical plan to identify the guilty wives. Gathering the men and women of the village together, he said that he was sorry to inform them that there had in fact been at least one instance of marital infidelity. He then instructed each husband to think carefully about his own wife. If, at any point, he concluded that she was guilty, then he was to take her in front of his house that same night, tie her to a tree and hang a red lantern over her head. The next morning she would be there for all to witness her shame.

At first the villagers all laughed. How was a husband supposed to discover his own wife's guilt? The next morning everyone was out to look. No red lanterns. Again the following morning, no red lanterns. This became a daily ritual. Although no red lanterns appeared, experience suggested that the High Priest knew what he was doing.

And he did. The fact was that in this village there were exactly *fifty guilty wives*. On the fiftieth morning, and no sooner, those fifty guilty wives, and no others, were standing in front of their fifty houses under fifty red lanterns!

We leave it to the reader to explain why this happened. It just might help to assume that these fifty betrayed husbands, cowardly male chauvinist pigs as they were, were nevertheless all graduates of the Consecutive Integer Academy.

Background and conclusions Our goal was to give some set of conditions under which the statement of Theorem 0 would be true. This is achieved in Theorem C. It remains to ask if there are any better, simpler, or more appropriate conditions that would serve the same purpose. In particular, are there conditions that allow a correct solution to the Consecutive Integer Game or the Fifty Guilty Wives Dilemma that is in fact a proper application of mathematical induction? It is worth noting that, in the process of establishing Theorems A, B and C, we never made any use of mathematical induction whatsoever!

I first contrived the Consecutive Integer Game as a means to introduce an undergraduate class to the concept of mathematical induction. I began with the argument preceding Theorem 0, and then led them through a conventional study of mathematical induction. It was only when I came back to give a formal proof of Theorem 0 that I realized I was in trouble. In a recent *Scientific American* article, Stewart [**4**] presents a problem that is mathematically identical to the Fifty Guilty Wives (though admittedly in much better taste!). There he describes the argument as "an instance of mathematical induction," and outlines a proof similar to our proof of Lemma A. Another variation is given by Myerson ([**3**, p. 66]) who also claims that it can be solved by mathematical induction. The analysis presented in this article would apply equally well to both of these problems.

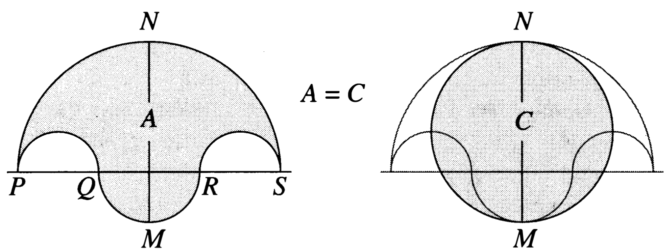
The Consecutive Integer Game shows us just how seriously we can go astray by not recognizing the difference between what is true and what a player knows to be true. Aumann [**1**] brought this distinction to focus in game theory by introducing the notion of **common knowledge**, that is, facts that are true, that each player knows to be true, that each player knows that each player knows to be true, and so forth. The relevance of common knowledge to these problems is articulated by Myerson [**3**]; see also Fudenberg and Tirole [**2**]. Our Theorem 0 is true provided not only that both players have the appropriate skills, but also that the fact they do is common knowledge. The CIA ∞–exam establishes the fact that they have these skills, but it is the CIA ∞–badges that make this fact common knowledge.

REFERENCES

1. R. J. Aumann, Agreeing to disagree, *Annals of Statistics* **4** (1976), 1236–1239.
2. D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1991.
3. R. Myerson, *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, MA, 1991.
4. I. Stewart, Mathematical recreations: Monks, blobs and common knowledge, *Scientific American* (August, 1998), 96–97.

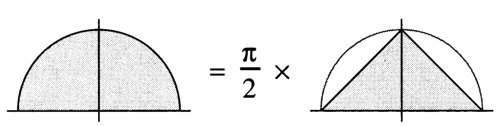
Proof Without Words: The Area of a Salinon

THEOREM. Let P, Q, R, S be four points on a line (in that order) such that $PQ = RS$. Semicircles are drawn above the line with diameters PQ, RS, and PS, and another semicircle with diameter QR is drawn below the line. A *salinon* is the figure bounded by these four semicircles. Let the axis of symmetry of the salinon intersect its boundary at M and N. Then the area A of the salinon equals the area C of the circle with diameter MN [Archimedes, *Liber Assumptorum*, Proposition 14].

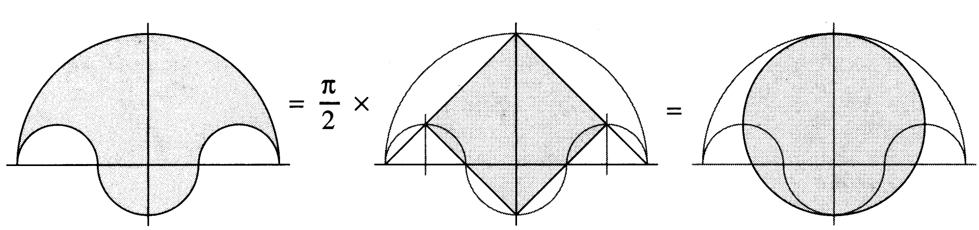


Proof.

I.



II.



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Asymptotic Symmetry of Polynomials

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We start with a simple visual observation about polynomials. Consider the polynomial

$$\begin{aligned} P(x) &= x^6 - 140x^5 + 8000x^4 - 238000x^3 + 3870000x^2 - 32400000x + 108000000 \\ &= (x-10)(x-20)^2(x-30)^3 \end{aligned}$$

and some plots of $y = P(x)$ at various scales.

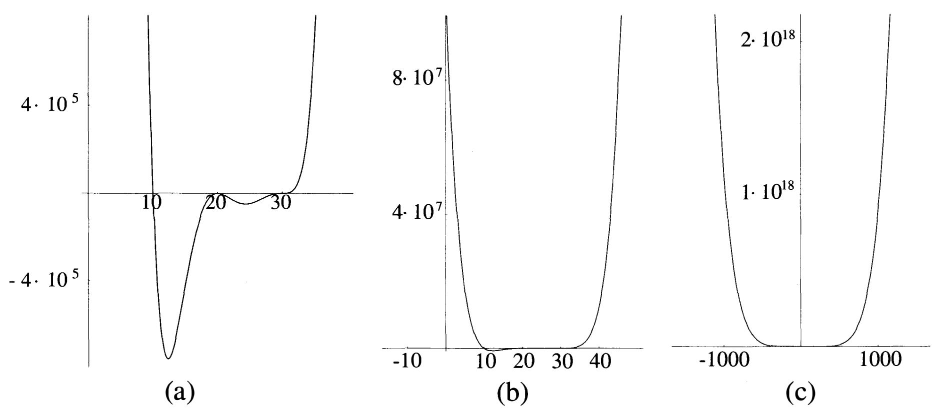
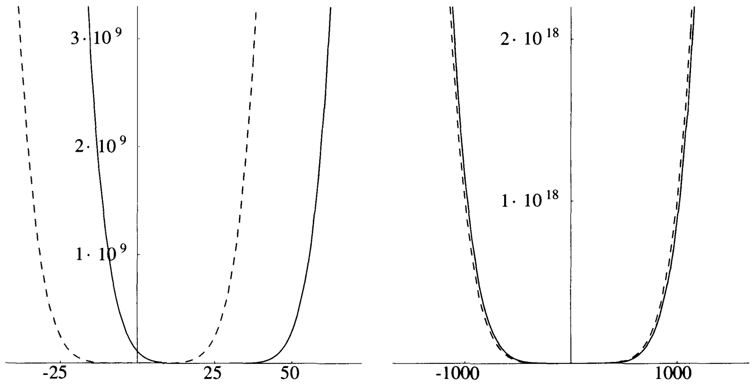


Figure 1 Several views of $P(x)$

In FIGURE 1(b) the graph appears roughly symmetric with respect to $x \approx 25$. In FIGURE 1(c), however, as we zoom out farther, this is no longer evident. In fact, the latter plot resembles that of $y = x^6$, which is the usual observation. What we address here is whether or not the *apparent* line of symmetry in FIGURE 1(b) is genuine, or fails to persist in any meaningful way as we zoom out to scales such as in FIGURE 1(c). Certainly we couldn't detect such an effect visually at such great distances.

A closer look from a distance When the absolute value of x is large, the graph of a polynomial begins to resemble that of the monomial consisting of the polynomial's leading term. Of course, this is justified by the observation that $P(x)$ is asymptotic to x^6. (By this we mean that $\lim_{x\to\pm\infty} P(x)/x^6 = 1$.) For the polynomial $P(x)$ above, this behavior is exhibited by plotting $P(x)$ along with $B(x) = x^6$ in FIGURES 2(a) and 2(b). The graphs of the two functions appear to merge as we zoom out until there is no apparent difference.

Obviously, zooming out far enough would result in the two graphs being indistinguishable, and that would be the case even if we let $B(x)$ be *any* polynomial with the



(a) (b)

Figure 2 The graphs of P (solid) and B (dotted) from afar

same leading term as $P(x)$. But in the distance chosen in FIGURE 2(a) the graph of P appears decidedly offset, horizontally, relative to B. So the question is, exactly what is this apparent horizontal offset? We might approach it this way: Is there a choice of h so that the graph of $(x-h)^6$ looks *most* like that of $P(x)$? However, since the graph of $(x-h)^6$ is symmetric with respect to the line $x = h$, we are more interested in asking: For what h is the line $x = h$ the axis of asymptotic symmetry of the graph of P? It turns out that for this example, most reasonable interpretations (some presented below) of either of these questions yield the same result, $h = 140/6$. See FIGURE 3.

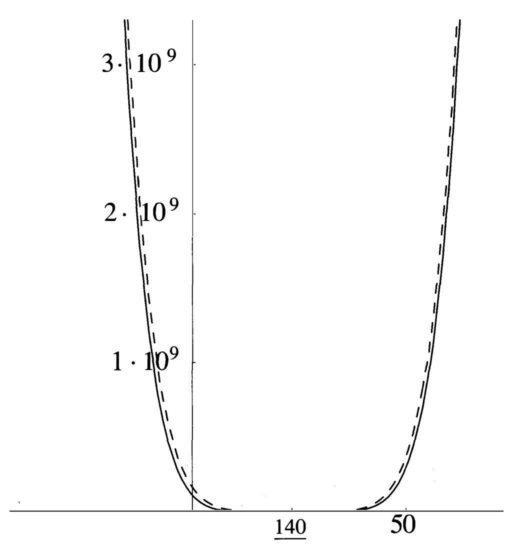


Figure 3 A plot of $P(x)$ (solid) along with the function $C(x) = (x - 140/6)^6$ (dashed)

We shall henceforth assume that $P(x)$ has even degree $n \geq 2$, and, for convenience, that its coefficients are real, with the leading coefficient being unity.

Before we can find the line of asymptotic symmetry, it would be nice if we defined it. One approach would be to choose h so that $P(x)$ and $P(2h - x)$ differ by an amount considered small as $x \to \pm\infty$. For instance, one could require h to satisfy

$$\lim_{x\to\pm\infty}\left[\frac{P(x)-P(2h-x)}{x^{n-1}}\right]=0.$$

Another possibility would be to choose h so $P(x)\approx C(x)=(x-h)^n$. For example, one would require h such that

$$\lim_{x\to\pm\infty}\left[\frac{P(x)-C(x)}{x^{n-1}}\right]=0.$$

(Notice that the usual approximating polynomial $y=x^n$ would not, in general, satisfy the above condition unless the power of x in the denominator was increased by one, and in this case, *any* h would do.)

However, both of these tentative definitions involve a specific power of x in the denominator, hence would not be suitable for generalization to nonpolynomial functions. Our solution is to replace the limit in the domain with a limit in the range, in a way inverting the first potential definition above. We fix the (large) y-coordinate and ask that the corresponding x-coordinates be asymptotically symmetric with respect to some line $x=h$, in the sense described below.

Assume that $\lim_{x\to\pm\infty} f(x)=+\infty$ and that $f(x)$ is eventually monotonic as $x\to\pm\infty$. For k big enough, this will imply that the horizontal line $y=k$ and the graph of f will have exactly two points of intersection, say, $(z_-(k),k)$ and $(z_+(k),k)$. We suppress the dependence upon k in what follows.

DEFINITION. If

$$\lim_{k\to\infty}\frac{z_+ + z_-}{2}=h,$$

we say $x=h$ is the line of asymptotic symmetry of the graph of f.

In the following theorem we apply this definition to find the line of asymptotic symmetry for polynomials of positive, even degree.

THEOREM. Let $P(x)=\prod_{i=1}^{n}(x-a_i)$, where the a_is may repeat or may occur as pairs of complex conjugates. Then the line $x=h$ is the line of asymptotic symmetry of P, provided that

$$h=\frac{1}{n}\sum_{i=1}^{n}a_i.$$

(Notice that h is the average of the roots of the polynomial, weighted by multiplicity.)

Proof. For $|x|$ large enough, P will be a locally one-to-one function whose local inverse has an asymptotic series expansion, valid near $\pm\infty$, being

$$P_\pm^{-1}(y)=\pm y^{1/n}+c_0\pm\frac{c_1}{y^{1/n}}+\frac{c_2}{y^{2/n}}\pm\frac{c_3}{y^{3/n}}+\frac{c_4}{y^{4/n}}\pm\cdots,$$

where the c_is are constants depending on P and $c_0=\frac{1}{n}\sum_{i=1}^{n}a_i$. (This expansion is justified in the appendix below.) This implies that

$$\begin{aligned}\frac{z_+ + z_-}{2}&=\frac{P_+^{-1}(k)+P_-^{-1}(k)}{2}\\&=c_0+\frac{c_2}{k^{2/n}}+\frac{c_4}{k^{4/n}}+\cdots,\end{aligned}$$

which approaches $c_0=h$ as $k\to\infty$, and the proof is complete. ∎

It should be noted that the three possible definitions for asymptotic symmetry proffered above might appear different, but (at least) for polynomials, using the asymptotic expression for the $z_{\pm}$, it is not hard to show all three are equivalent.

Generalizations We conclude this note with some suggested projects for students.

(a) What about asymptotic symmetry with respect to a point? For example, cubic polynomials have symmetry about their inflection points; do all polynomials of odd degree have a point of asymptotic symmetry?

(b) What would be the line (or point) of asymptotic symmetry for rational functions with degree of the numerator larger than the degree of the denominator, or more generally, functions that are written as products of linear factors with real coefficients when the exponents are not necessarily positive integers, so long as the factors are defined for all x and the degree of the resultant algebraic expression is positive? (In this case, the proof given above for polynomials of even degree can be easily modified to handle functions analogous to polynomials of positive, even degree, for instance, $y = x^{1/5}(x-1)^{2/3}/(x-2)^{1/7}$. See FIGURE 4 at the end of this note.) What happens if one allows other types of functions, e.g., $y = \sqrt{x^2 - x^{1/3}}$ or $y = x + \cosh(x)$?

(c) What problems arise if one drops the assumption that the function is eventually monotonic as $x \to \pm\infty$?

Appendix We now justify the expansion in the theorem for the local inverse of $P(x)$. We first find the local inverse for x near positive infinity. For $1 \leq k < n$, let b_k be the coefficient of x^k in $P(x)$. So, for example, $b_{n-1} = -nh$, where $h = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then

$$\begin{aligned} y = P(x) &= x^n + b_{n-1}x^{n-1} + \ldots + b_0 \\ &= x^n[1 + L(1/x)], \end{aligned}$$

where $L(1/x) = b_{n-1}/x + \ldots + b_0/x^n$. Let $u = +y^{1/n}$, which is a one-to-one substitution for x near positive infinity, so that $u = x[1 + L(1/x)]^{1/n}$.

Since $L(1/x)$ approaches zero as x approaches infinity, eventually $|L(1/x)| < 1$ for x large enough, so one can expand $[1 + L(1/x)]^{1/n}$ via the binomial theorem. The result is

$$\begin{aligned} u &= x\left[1 + (1/n)L(1/x) + \frac{(1/n)(1/n-1)}{2!}L(1/x)^2 + \cdots\right] \\ &= x[1 + (b_{n-1}/n)(1/x) + (\text{higher powers of } 1/x)] \\ &= x + b_{n-1}/n + (\text{higher powers of } 1/x). \end{aligned}$$

One now solves for the inverse function of the form

$$x = G(u) = u + c_0 + \frac{c_1}{u} + \cdots$$

by forming the equation $u = P(G(u))$ and recursively solving for the coefficients c_0, c_1, etc. In particular, $c_0 = -b_{n-1}/n = h$. The local expansion of the inverse is then given by $x = G\left(y^{1/n}\right)$.

For the local inverse when x is near minus infinity, the only change is that now u becomes $-y^{1/n}$, and the rest is identical.

Encore We cannot resist one more picture. Let

$$f(x) = x^{1/5}(x-1)^{2/3}/(x-2)^{1/7},$$

let $n = 76/105$, and let $h = 10/19$. Below are the graphs of $y = f(x)$ (solid) and $y = (x-h)^n$ (dashed).

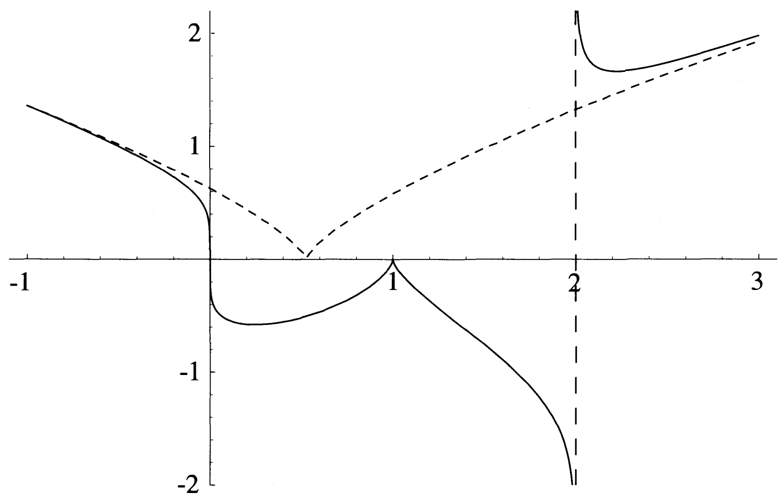


Figure 4 A more general example

Duality and Symmetry in the Hypergeometric Distribution

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A CNN posting on the internet [**1**] reports that the 1996 trial of a rap star on manslaughter charges resulted in a hung jury. The posting says that the jury was composed of 7 men and 5 women and was hung at 9 to 3. Not reported is how many men or women voted with the majority. Several interesting probability problems come to mind. For example, what is the probability that exactly 5 jurors among the majority are men?

The usual solution for such a problem utilizes methods that are associated with the hypergeometric probability distribution and involves designating successes and choosing a sample from a population. For the problem stated, it is clear that the population is the 12 jurors. But for the men and the majority, it is not so clear which should be considered the successes and which the sample; either way seems artificial and contrived. We solve this problem later, in a more natural way, using a symmetric form of the hypergeometric distribution derived in this note.

Using the classical terminology, as we have, for describing the hypergeometric distribution, we consider a population of size n, a number i of which are specified as successes. Our probability experiment is the random choice of a sample of size j from the population. We would like to know the probability that the random variable X, which is the number of successes among the sample, has value x. This is given by the hypergeometric distribution, specifically,

$$P(X = x) = \binom{i}{x}\binom{n-i}{j-x} \Big/ \binom{n}{j}$$

when $0, i + j - n \leq x \leq i, j$.

The hypergeometric distribution admits what may be called *dual* and *symmetric* forms. In the dual formulation, we can interchange the sample with the successes and arrive at the equally valid

$$P(X = x) = \binom{j}{x}\binom{n-j}{i-x} \Big/ \binom{n}{i}.$$

This duality has been brought out in these Notes by Davidson and Johnson [**2**], is mentioned by Feller [**3**, page 44], and is easily verified by expanding all the binomial coefficients into factorials. Common examples where duality emerges naturally include lottery games [**4**, Exercises 9 and 10, page 172] and selections by lot. Here is an instance of the latter.

Three persons are chosen by lot from a group of 10. What is the probability that a specific person from the group is chosen?

Here, a sample of 3 is chosen from a population of 10 that includes 1 success, the specific person. Hence, the probability is $P(X = 1) = \binom{1}{1}\binom{9}{2}/\binom{10}{3} = \frac{3}{10}$. Dually, the specific person, the sample, has 3 chances, the successes, out of 10 of being chosen. Thus, $P(X = 1) = \frac{3}{10} = \binom{3}{1}\binom{7}{0}/\binom{10}{1}$.

Returning to our beginning example, we see that the way the men and majority are regarded as the successes and sample does not matter, and once a decision is made, the probability is easily determined. However, having no grounds to decide, we prefer to respect the symmetry in the problem.

By treating the successes and the sample symmetrically, we derive another formula for the hypergeometric distribution. Consider the population as a set U in which the members are *independently* classified in two ways yielding the subsets I and J of successes and of sample points, respectively. Then X gives the size of the intersection $I \cap J$. We note that I and J are combinations of sizes i and j chosen from U. To find $P(X = x)$, we consider all such pairs (I, J) of these combinations as equally likely outcomes. The size of the sample space of all these outcomes is

$$\binom{n}{i}\binom{n}{j}.$$

The event that $I \cap J$ has size x corresponds to all the ways of partitioning U into four subsets $I \cap J$, $I \cap \bar{J}$, $\bar{I} \cap J$, and $\bar{I} \cap \bar{J}$ of sizes x, $i - x$, $j - x$, and $n - i - j + x$, respectively (FIGURE 1). Recall that the multinomial coefficient

$$\binom{n}{n_1, n_2, n_3, n_4} = \frac{n!}{n_1!\, n_2!\, n_3!\, n_4!}$$

U	J	$\bar{J}$	
I	x	$i-x$	i
$\bar{I}$	$j-x$	$n-i-j+x$	$n-i$
	j	$n-j$	n

Figure 1 Partition of U

counts the number of ways to partition a set of n elements into 4 disjoint subsets with n_k elements in the kth part. It is ideal for counting the possible partitions of the kind indicated in the diagram. There are

$$\binom{n}{x,\ i-x,\ j-x,\ n-i-j+x}$$

such partitions. Therefore, the probability that $I \cap J$ has size x is equal to

$$P(X=x) = \binom{n}{x,\ i-x,\ j-x,\ n-i-j+x} \Big/ \binom{n}{i}\binom{n}{j}.$$

The use of this symmetric form of the hypergeometric distribution seems more natural than either of the dual ones in problems where no sample is chosen. Our opening example is such a problem.

We assume that being a man and being among the majority are independent events. Let U be the group of jurors, I the subgroup of men, and J the subgroup of the majority. Therefore, the probability that exactly 5 jurors among the majority are men is

$$P(X=5) = \binom{12}{5,\ 2,\ 4,\ 1} \Big/ \binom{12}{7}\binom{12}{9} = \frac{21}{44}.$$

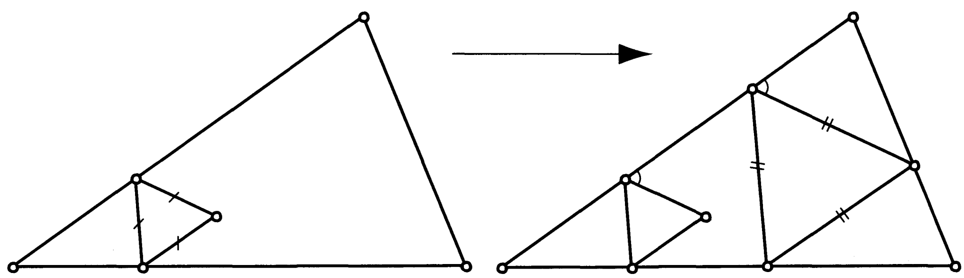
Here is another symmetric problem in which choosing a sample and designating successes would be artificial. A coed softball team of 17 players includes 8 females and 14 players who throw right-handed. Assuming independence of these traits, what is the probability that exactly 6 players are right-handed-throwing females? The solution can be found on page 143.

Acknowledgment. We thank the anonymous referees for their useful suggestions.

REFERENCES

1. CNN, *Hung jury in rap star's manslaughter trial*, http://www.cnn.com/US/9602/snoop.
2. R. R. Davidson and B. R. Johnson, Interchanging parameters of the hypergeometric distribution, this MAGAZINE **66** (1993), 328–329.
3. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd Ed., John Wiley, New York, 1968.
4. H. J. Larson, *Introduction to Probability*, Addison-Wesley, New York, 1995.

Proof Without Words: Every Triangle Has Infinitely Many Inscribed Equilateral Triangles



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Perfect Cyclic Quadrilaterals

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Are there any quadrilaterals with integer sides having perimeter P equal to area A? A square of side length 4 might come to mind. Are there any more? More generally, what is the number $N(k)$ of integer-sided quadrilaterals whose ratio of perimeter to area is a fixed value, say $P/A = k$? This question is interesting mainly for cyclic quadrilaterals (that is, those that can be inscribed in a circle) since there are, for example, an infinite number of parallelograms satisfying $P = kA$ for a given positive number k (as the reader can check). In addition, the cyclic case generalizes the question for triangles, which has been treated successfully.

In 1971, M. V. Subbarao [**6**] considered the problem of finding the number $N(\lambda)$ of triangles for which the sum of the integer sides a, b, and c is equal to λ times the triangle's area, where λ is a given positive real number. These are called *perfect triangles*. He showed that $N(\lambda)$ is finite with $N(\lambda) = 0$ for $\lambda > \sqrt{8}$, $\lambda \neq 2\sqrt{3}$, and $N(\lambda) = 1$ for $\lambda = 2\sqrt{3}$, the triangle being equilateral with edge 2. Although Subbarao does not say so explicitly, the integers a, b, c are presumed to be greater than 1 since any triangle with sides $1, a, a$ has $\lambda > 4$, as the reader may enjoy showing.

Previous authors have considered special cases. In 1904, Whitworth and Biddle [**2**] showed that the only triangles with perimeter equal to the area are the (5, 12, 13), (6, 8, 10), (6, 25, 29), (7, 15, 20), and (9, 10, 17) triangles. In 1955, Fine [**3**] settled the case when $\lambda = 2$, answering a question posed by Phelps [**5**], namely that the (3, 4, 5) triangle is the only one whose perimeter is twice the area. In his closing remarks, Subbarao asked whether there is an analogy for quadrilaterals.

The purpose of this paper is to extend these results and discuss the number $N(k)$ of cyclic quadrilaterals with integer sides (including 1) satisfying $P = kA$, where k is a positive real number, and P and A are the perimeter and area of a quadrilateral. These quadrilaterals are said to be *k-perfect*. We will show that $N(k)$ is finite with $N(k) = 0$ for $k > 4$. Furthermore, when k is an integer we have $N(1) = 7$, $N(3) = 2$, and $N(2) = N(4) = 1$.

Eliminating $P/A > 4$ The area of a convex quadrilateral having side lengths a, b, c, d is given by

$$A^2 = (s-a)(s-b)(s-c)(s-d) - (1/2)abcd\,(1 + \cos(\theta + \lambda)),$$

where θ and λ are opposite interior angles and s is the semi-perimeter $s = (a + b + c + d)/2$ [**4**]. If the quadrilateral is cyclic then $\theta + \lambda = 180°$ (see [**1**], p. 127) so the area equation reduces to Brahmagupta's formula $A^2 = (s-a)(s-b)(s-c)(s-d)$. Letting $M_1 = -a + b + c + d$, $M_2 = a - b + c + d$, $M_3 = a + b - c + d$, and $M_4 = a + b + c - d$ this equation may be written

$$\begin{aligned} 16A^2 &= M_1 M_2 M_3 M_4 \\ &= (-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d). \quad (1) \end{aligned}$$

Let's assume $a \geq b \geq c \geq d$. Equation (1) shows that

$$\begin{aligned} 16A^2 &\geq M_1(c + d)(a + d)(a + b) = M_1(a + b)(ac + ad + cd + d^2) \\ &\geq M_1(a + b)(bd + ad + cd + d^2) = M_1 d(a + b)P. \quad (2) \end{aligned}$$

Inequality (2) reduces to an equality if and only if $a = b = c = d$. This is the square case where $k = P/A = 4a/a^2 = 4/a$. Thus each square (with an integer side) is a k-perfect quadrilateral with $k \leq 4$. If we exclude the square case then $16A^2 > M_1 d(a + b)P$, which yields

$$\begin{aligned} k^2 = P^2/A^2 &< \frac{16P^2}{M_1 d(a + b)P} = \frac{16P}{M_1 d(a + b)} \\ &= \left(\frac{16}{M_1 d}\right)\left(1 + \frac{c + d}{a + b}\right) \leq 32/M_1 d. \quad (3) \end{aligned}$$

If $M_1 d \geq 2$ then $k \leq 4$. If $M_1 d = 1$ then $M_1 = d = 1$, so $P = M_1 + 2a = 2a + 1$ and $b + c = a$. Thus

$$\begin{aligned} k^2 = P^2/A^2 &= \frac{16P^2}{M_1 M_2 M_3 M_4} = \frac{16(2a + 1)^2}{(2c + 1)(2b + 1)(2a - 1)} \\ &= \frac{16(2a + 1)^2}{(4bc + 2a + 1)(2a - 1)} \leq \frac{16(2a + 1)^2}{(4a + 1)(2a - 1)} \leq 16, \end{aligned}$$

because $a \geq 2$ (in the nonsquare case). This shows that $k \leq 4$ in all cases, so $N(k) = 0$ for $k > 4$.

The few cases with $3 \le P/A \le 4$ Assume for the moment that $c = d = 1$. Then $16A^2 = (-a+b+2)(a-b+2)(a+b)^2$ from (1) and $P = a+b+2$, so

$$k^2 = P^2/A^2 = \frac{16(a+b+2)^2}{(a+b)^2\left[4-(a-b)^2\right]}.$$

Since $4-(a-b)^2 > 0$ there are two cases to consider. If $a = b$ then we have $k = 2(b+1)/b$. This shows that $k < 3$ if $b \ge 3$. When $b = 1$ we have a square (with $k = 4$) and when $b = 2$, we obtain two noncongruent quadrilaterals with sides 2, 2, 1, 1 (and $k = 3$): one when equal sides are opposite and one when they are not. If $a = b+1$ then $k^2 = 16(2b+3)^2/(3(2b+1)^2)$, which implies that $k < 3$ if $b \ge 3$. When $b = 2$ we obtain two noncongruent quadrilaterals with sides 3, 2, 1, 1 and $k = 28/5\sqrt{3}$, and when $b = 1$ we obtain a quadrilateral with sides 2, 1, 1, 1 and $k = 20/3\sqrt{3}$.

We can show that these are the only perfect quadrilaterals when $P/A \ge 3$. That is, $N(k) = 0$ for $k \ge 3$ except for the values $k = 3, 28/5\sqrt{3}, 20/3\sqrt{3}$, and 4. This can be done by examining the values of $M_1 d$ in (3). If $M_1 d \ge 4$ then (3) shows that $k < 3$ so we should examine the choices $M_1 d = 1, 2, 3$.

Suppose that $M_1 d = 1$. Then $M_1 = d = 1$. Having just dealt with the possibilities when $c = 1$, we may assume that $c \ge 2$. We then have $bc \ge b + c$, and the computation at the close of the preceding section takes the form

$$k^2 = \frac{16(2a+1)^2}{(4bc+2a+1)(2a-1)} \le \frac{16(2a+1)^2}{(6a+1)(2a-1)} = \frac{16(4a^2+4a+1)}{12a^2-4a-1} < 9$$

when $a \ge 3$. We have already treated the cases with $a \le 2$ except for the quadrilateral with sides 2, 2, 2, 1. But this case has $M_1 d = 3$ and $2 < P/A < 3$. Thus $k = P/A < 3$ except for the cases above.

Now suppose that $M_1 d = 2$. Then $M_1 = 1$ and $d = 2$ or $M_1 = 2$ and $d = 1$.

Case 1: $M_1 = 1$ implies that $P = M_1 + 2a = 1 + 2a$ and $b + c = a - 1$. Since $d = 2$, $bc \ge b + c$ and $P \ge 8$ so $a \ge 4$. Thus (1) becomes

$$16A^2 = (1)(2c+3)(2b+3)(2b+2c-1)$$

so

$$\begin{aligned} P^2/A^2 &= \frac{16(2a+1)^2}{(4bc+6(b+c)+9)(2b+2c-1)} \\ &\le \frac{16(2a+1)^2}{(4(a-1)+6(a-1)+9)(2a-3)} = \frac{16(2a+1)^2}{(10a-1)(2a-3)} < 9, \end{aligned}$$

since $a \ge 4$.

Case 2: $M_1 = 2$ implies that $P = M_1 + 2a = 2 + 2a$ and $b + c = a + 1$. Thus

$$16A^2 = (2)(2c)(2b)(2b+2c-2),$$

so

$$P^2/A^2 = \frac{16(2a+2)^2}{(16bc)(b+c-1)} \le \frac{(2a+2)^2}{(b+c)(b+c-1)} = \frac{4(a+1)^2}{(a+1)a} < 9.$$

Thus $k < 3$ in both cases. Similar reasoning establishes this inequality when $M_1 d = 3$.

We conclude that for $k \ge 3$, $N(k) = 0$ except for the few values mentioned above. In contrast note that for $2 < k < 3$, $N(k)$ is positive for infinitely many values k as illustrated by the quadrilaterals with sides $a, a, 1, 1$, where $a > 2$ is an integer.

When the perimeter-area ratio is an integer In this section, we assume that k is an integer; we will return to the general case in the next section. If $d \geq 32$ then (3) shows that k is not an integer, so we may assume $d < 32$. A short computer run (sides up to 50) suggests that the only cases for which P/A is an integer are those shown in TABLE 1. We will explain why this is indeed so. In each case where only two of the sides are equal, we obtain two noncongruent quadrilaterals: one when the equal sides are opposite and one when they are not. Thus $N(1) = 7$, $N(2) = N(4) = 1$, and $N(3) = 2$.

TABLE 1: The k-perfect quadrilaterals when k is an integer

$k = 4,$	$a = b = c = d = 1$
$k = 3,$	$a = b = 2, \quad c = d = 1$
$k = 2,$	$a = b = c = d = 2$
$k = 1,$	$a = b = c = d = 4$
$k = 1,$	$a = b = 6, \quad c = d = 3$
$k = 1,$	$a = 8, \quad b = c = 5, \quad d = 2$
$k = 1,$	$a = 14, \quad b = 6, \quad c = d = 5.$

Let us see why TABLE 1 is complete. Actually, the case where $d = 1$ is easily handled. Writing $M_4 = P - 2$, $m = M_1M_2M_3$, and setting $P = kA$ we obtain (from equation (1)),

$$16P^2 = k^2(P - 2)m, \tag{4}$$

showing $P - 2$ to be a power of 2 (an odd prime factor of $P - 2$ would divide P). Hence P is even (so each M_i is even also) and $P/2$ is odd. The left-hand side of (4) contains exactly six factors of 2; thus $P - 2$ is either 2, 4, or 8 (since m is a multiple of 8). If $P = 4$ or 6, then $a \leq 2$, and these cases were described in the preceding section. If $P = 10$ then $M_1 = P - 2a = 10 - 2a$ so $M_1/2 = 5 - a$ forcing $a \leq 4$, and these few cases are easily ruled out.

Thus for the remainder of this section we assume that $d \geq 2$. Then (3) shows that $k < 4$. Now we can see that P is even (when k is an integer). For if we assume that P is odd then so is each M_i, so (1) shows $16P^2 = k^2M_1M_2M_3M_4$ giving $k \equiv 0 \pmod 4$. This contradicts the fact that $k < 4$. Thus P and each M_i are even. In particular, $M_1 \geq 2$ so $k < \sqrt{32/M_1d} \leq 4/\sqrt{d}$. Since $k \geq 1$ we have $d < 16$. Also note that when $d \geq 4$ we have $k = 1$.

We will examine the cases $d \leq 15$. Our plan is to find an upper bound for the perimeter in each case. A computer run can then produce any perfect quadrilateral. Writing $P = kA$ and using (1) we obtain

$$16P^2 = k^2(P - 2d)m, \tag{5}$$

where $m = M_1M_2M_3$.

If $d = 2$, then similar reasoning surrounding (4) shows that, since $16P^2 = k^2(P - 4)m$, $P - 4$ is a power of 2. Since $P \geq 8$ we have $P = 4 + 2^i$ with $i \geq 2$. If $i \geq 3$ then $i \leq 5$ since $P = 4(4 + 2^{i-2})$, showing $P/4$ to be odd; this gives at most eight factors of 2 on the left-hand side of (5). Since m is a multiple of 8, $P - 4$ contains at most five factors of 2. We obtain $P = 8, 12, 20, 36$ as the possibilities, and TABLE 1 shows the outcome. Similarly, we can handle $d = 4$ giving possible values $P = 16, 24, 40, 72, 136$. If $d = 8$, then $P = 16 + 2^i$, for $i = 4, 5, 6, 7, 8, 9$ so $P \leq 528$.

Suppose $d = 3$. Then equation (5) becomes

$$16P^2 = k^2(P-6)m, \tag{6}$$

so the only primes dividing $P-6$ are 2 and 3. We have $P-6 = 2^i3^j$, where $i \geq 1$, $j \geq 0$. Writing $P = 6 + 2^i3^j$ we see that $j \leq 2$, since if 27 divides $P-6$ then 9 divides P, but $P = 3(2 + 2^i3^{j-1})$ cannot contain two factors of 3. Also, if $i > 1$ then $i \leq 3$ since $P = 2(3 + 2^{i-1}3^j)$, showing $P/2$ to be odd; this gives at most six factors of 2 on the left-hand side of (6), so $P-6$ contains at most three factors of 2. Thus $P = 6 + 2^i3^j$ with $i \leq 3$ and $j \leq 2$, so the maximum possible value of P is 78.

A similar argument applies if d is any odd prime. For example, if $d = 5$ then $P = 10 + 2^i5^j$ with $i \leq 3$ and $j \leq 2$, so $P \leq 210$. The upper bounds for $d = 7$, 11, and 13 are 406, 990, and 1378, respectively.

Let us deal with the case with $d = 2q$, where q is an odd prime. Equation (5) becomes

$$16P^2 = (P-4q)m, \tag{7}$$

so the only primes dividing $P-4q$ are 2 and q. We have $P-4q = 2^iq^j$, where $i \geq 1$, $j \geq 0$. Writing $P = 4q + 2^iq^j$, we see that $j \leq 2$, since if q^3 divides $P-4q$ then q^2 divides P, but $P = q(4 + 2^iq^{j-1})$ cannot contain two factors of q. Also, if $i \geq 3$ then $i \leq 5$ since $P = 4(q + 2^{i-2}q^j)$, showing $P/4$ to be odd; this gives at most eight factors of 2 on the left-hand side of (7), so $P-4q$ contains at most five factors of 2. This shows that $P = 4q + 2^iq^j$ with $i \leq 5$ and $j \leq 2$. When $d = 6$ ($q = 3$), the maximum possible value of P is 300. If $d = 10$ the maximum is 820, and when $d = 14$ the maximum is 1596.

Similar reasoning shows that if $d = 9$, then $P = 18 + 2^i3^j$, where $i \leq 3$, $j \leq 4$, and the largest of these is 666. If $d = 12$, then $P = 24 + 2^i3^j$, where $i \leq 7$ and $j \leq 2$, giving the upper bound 1176. If $d = 15$, then $P = 30 + 2^i3^j5^t$, with a maximum of 1830.

These upper bounds for P are summarized in TABLE 2. Since $a < b + c + d$, we have $a < P/2$, which gives an upper bound for a in any computer run. Since $k \geq 1$, (3) indicates that $M_1d < 32$; for $d \geq 4$ we can then obtain the upper bound $c \leq P/4$ as follows: We have $P - 2d + M_1 = M_4 + M_1 = 2(b+c)$, so

$$2c \leq b + c = (P - 2d + M_1)/2 \leq P/2.$$

Note that $(-2d + M_1)d = -2d^2 + M_1d < -2d^2 + 32 \leq 0$, so $-2d + M_1 \leq 0$ if $d \geq 4$. Our computer run found that for $d \leq 15$, the only perfect quadrilaterals are those shown in TABLE 1.

TABLE 2: Upper bounds for P

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P	10	36	78	136	210	300	406	528	666	820	990	1176	1378	1596	1830

The general case Our work has shown how the perimeter P of a k-perfect quadrilateral is bounded when $k = P/A$ is an integer. For the general case, equation (5) shows that k^2 is rational. Let $k^2 = r/s$ where r and s are integers. We can find an upper bound on P by reasoning as before. Recall from (3) that $k^2 \leq 32/d$, so $d \leq 32/k^2$; in the square case $d = 4/k$. Thus d is bounded. Equation (5) becomes

$$16sP^2 = r(P-2d)m. \tag{8}$$

We see that any prime divisor of $P - 2d$ that does not divide $16s$ must divide P and hence d. Thus, the number of distinct primes of $P - 2d$ is limited to those of $2s$ and d. As in the earlier cases, (8) shows that the multiplicity of such a prime is bounded. [Let $d = q^i u$ and $P - 2d = q^j v$, where q is a prime, i and j are maximal, and $j > i$. Then $P = q^i(2u - q^{j-i}v)$. Equation (8) shows that j can not exceed $2i$ + the number of factors of q in $16s$.] For a given value k, $P - 2d$ (and hence P) is bounded. This shows that $N(k)$ is finite.

REFERENCES

1. N. Altshiller-Court, *College Geometry*, 2nd ed., Barnes & Noble, New York, 1952.
2. L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, New York, 1952.
3. N. J. Fine, Solution to a problem of Phelps, *Amer. Math. Monthly* **63** (1956), 43–44.
4. I. Niven, *Maxima and Minima without Calculus*, Math. Assoc. of America, Washington, DC, 1981.
5. R. P. Phelps, Problem posed by, *Amer. Math. Monthly* **62** (1955), 365.
6. M. V. Subbarao, Perfect triangles, *Amer. Math. Monthly* **72** (1971), 384.

A Hypergeometric Problem Solved (from p. 137)

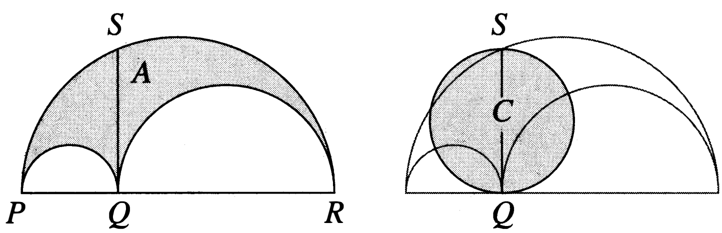
A coed softball team of 17 players includes 8 females and 14 players who throw right-handed. Assuming independence of these traits, what is the probability that exactly 6 players are right-handed-throwing females?

Solution.

$$\binom{17}{6,\ 2,\ 8,\ 1} \Big/ \binom{17}{8}\binom{17}{14} = \frac{63}{170}.$$

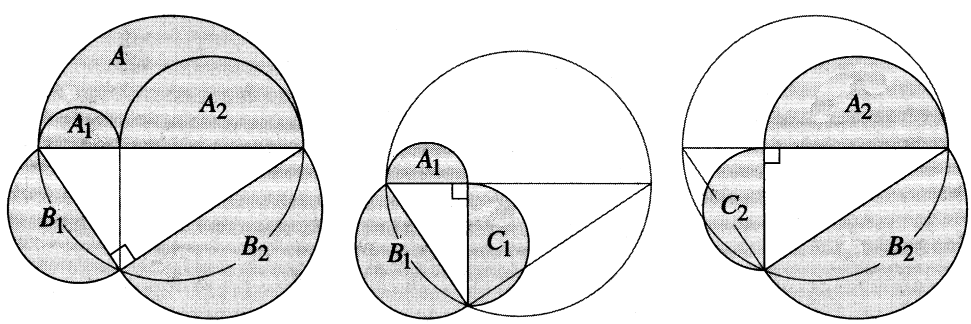
Proof Without Words: The Area of an Arbelos

THEOREM. Let P, Q, and R be three points on a line, with Q lying between P and R. Semicircles are drawn on the same side of the line with with diameters PQ, QR, and PR. An *arbelos* is the figure bounded by these three semicircles. Draw the perpendicular to PR at Q, meeting the largest semicircle at S. Then the area A of the arbelos equals the area C of the circle with diameter QS [Archimedes, *Liber Assumptorum*, Proposition 4].



$A = C$

Proof.



$A + A_1 + A_2 = B_1 + B_2$ $\quad$ $B_1 = A_1 + C_1$ $\quad$ $B_2 + A_2 + C_2$

$$A + A_1 + A_2 = A_1 + C_1 + A_2 + C_2$$

$$\therefore A = C_1 + C_2 = C$$

——ROGER B. NELSEN
LEWIS & CLARK COLLEGE

PROBLEMS

ELGIN H. JOHNSTON, *Editor*
Iowa State University

Assistant Editors: RĂZVAN GELCA, Texas Tech University; ROBERT GREGORAC, Iowa State University; GERALD HEUER, Concordia College; VANIA MASCIONI, Western Washington University; PAUL ZEITZ, The University of San Francisco

Proposals

To be considered for publication, solutions should be received by September 1, 2002.

1643. *Proposed by Árpád Bényi, University of Kansas, Lawrence, KS, and Ioan Caşu, West University of Timişoara, Timişoara, Romania.*

The sequence $(x_n)_{n\geq 0}$ of nonnegative real numbers satisfies the inequalities

$$x_{n-1}^2 \leq c x_{n-2} x_n, \qquad n \geq 2,$$

where c is a positive constant. Show that for integers n and k, with $0 \leq k \leq n$,

$$x_k \leq c^{k(n-k)/2} x_n^{k/n} x_0^{(n-k)/n}.$$

1644. *Proposed by Michael Golomb, Purdue University, West Lafayette, IN.*

Assume that the continuous, real valued functions f_i, $i = 1, 2$, are defined on the domain $\mathcal{D} = \{(x, y) : 0 \leq x \leq y \leq 1\}$ and satisfy the following:

(1) $f_i(x, x) = 0, \qquad 0 \leq x \leq 1,$
(2) $f_i(0, x) + f_i(x, 1) = 1, \qquad 0 \leq x \leq 1,$
(3) $f_i(x, y)$ is strictly decreasing in x and strictly increasing in y.

Show that there is a point $(x_0, y_0) \in \mathcal{D}$ such that $f_1(x_0, y_0) = f_2(x_0, y_0) = \frac{1}{2}$.

1645. *Proposed by Philip Straffin, Stephen Goodloe, and Tamas Varga, Beloit College, Beloit, WI.*

A graph is called *magic* if it has $n \geq 1$ edges and its edges can be labeled by the integers $1, 2, \ldots, n$ with each integer used once, and so that the sum of the labels of the edges at any vertex is the same. Are there any magic graphs that are not connected?

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Elgin Johnston, Problems Editor, Department of Mathematics, Iowa State University, Ames, IA 50011, or mailed electronically (ideally as a LaTeX file) to ehjohnst@iastate.edu. All communications should include the readers name, full address, and an e-mail address and/or FAX number.

1646. *Proposed by Erwin Just (Emeritus), Bronx Community College, New York, NY.*

Let $a > 0$, b, $k > 0$, and $m > 0$ be integers, and assume that the arithmetic progression $\{an+b\}_{n=0}^{\infty}$ contains the kth power of an integer. Prove that there are an infinite number of values of n for which $an+b$ is the sum of m kth powers of nonzero integers.

1647. *Proposed by Leroy Quet, Denver, CO.*

Prove that for all $x > 0$,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{(j+1)^{-2}}-x^{k^{-2}}}{(j+1)^2-k^2}=\frac{\pi^2}{8}-\frac{3}{4}\sum_{j=1}^{\infty}\frac{x^{j^{-2}}}{j^2}.$$

Quickies

Answers to the Quickies are on page 151.

Q919. *Proposed by Răzvan Gelca, Texas Tech University, Lubbuck, TX.*

Let ABC be a right triangle with right angle at A. On BC construct equilateral triangle BCD exterior to ABC. Prove that the lengths of AB, AC, and AD cannot all be rational numbers.

Q920. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada.*

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is inscribed in a parallelogram. Determine the area of the parallelogram if two of the points of tangency are $(a\cos u, b\sin u)$ and $(a\cos v, b\sin v)$, with $0 \le u < v < \pi$.

Solutions

Rational Bounds for the Sine Function **April 2001**

1618. *Proposed by Michael Golomb, Purdue University, West Lafayette, IN.*

Prove that for $0 < x < \pi$,

$$x\frac{\pi-x}{\pi+x} < \sin x < \left(3-\frac{x}{\pi}\right)x\frac{\pi-x}{\pi+x}.$$

Solution by Tom Jager, Calvin College, Grand Rapids, MI.

We prove the stronger inequality

$$\frac{x(\pi-x)}{\pi} < \sin x < \frac{x(\pi-x)(2\pi-x)}{\pi^2}, \qquad 0 < x < \pi. \qquad (*)$$

We prove the left inequality first. Using the Maclaurin series for $\sin x$ we have

$$\begin{aligned}\sin x - \frac{x(\pi-x)}{\pi} &= \left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)-x+\frac{x^2}{\pi}\\ &= x^2\left(\frac{1}{\pi}-\frac{x}{6}\right)+\sum_{k=1}^{\infty}\frac{x^{4k+1}}{(4k+1)!}\left(1-\frac{x^2}{(4k+2)(4k+3)}\right).\end{aligned}$$

For $0 < x \le \pi/2$, it is clear that this expression is positive. Because $\sin x - \frac{x(\pi - x)}{\pi}$ is symmetric about $x = \pi/2$, it follows that the left inequality in $(*)$ holds.

We next prove the right inequality. Using the Taylor expansion about π for $\sin x$ we obtain

$$\begin{aligned}
\sin x &- \frac{x(\pi - x)(2\pi - x)}{\pi^2} \\
&= \left(\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x-\pi)^{2k+1}}{(2k+1)!}\right) - (\pi - x)\left(1 - \frac{(x-\pi)^2}{\pi^2}\right) \\
&= (\pi - x)^3\left(\frac{1}{\pi^2} - \frac{1}{6} + \frac{1}{120}(x-\pi)^2\right) \\
&\quad - \sum_{k=2}^{\infty}\frac{1}{(4k-1)!}(\pi - x)^{4k-1}\left(1 - \frac{(x-\pi)^2}{4k(4k+1)}\right).
\end{aligned}$$

If $1 \le x < \pi$ then $\frac{1}{\pi^2} - \frac{1}{6} + \frac{1}{120}(x - \pi)^2 < 0$ proving that the right inequality in $(*)$ holds for such x. To prove the inequality for $0 < x \le 1$ use the Maclaurin series for $\sin x$ to see

$$\begin{aligned}
\sin x &- \frac{x(\pi - x)(2\pi - x)}{\pi^2} \\
&= -x + \frac{3}{\pi}x^2 - \left(\frac{1}{3!} + \frac{1}{\pi^2}\right)x^3 + \frac{x^5}{5!} - \sum_{k=2}^{\infty}\frac{x^{4k-1}}{(4k-1)!}\left(1 - \frac{x^2}{4k(4k+1)}\right) \\
&< x\left(\left(-1 + \frac{3}{\pi}x\right) + x^2\left(-\frac{1}{6} - \frac{1}{\pi^2} + \frac{x^2}{120}\right)\right) < 0
\end{aligned}$$

for x in this range. This completes the proof of the right inequality in $(*)$.

Also solved by Enhand Araüne (Austria), Paul Bracken (Canada), Daniele Donini (Italy), Robert L. Doucette, Ovidiu Furdui, Stephen Kaczkowski, Phil McCartney, Heinz-Jürgen Seiffert (Germany), Ajaj A. Tarabay and Bassem B. Ghalayini (Lebanon), Problem Solving Seminar at UAB, Xianfu Wang (Canada), Michael Woltermann, Li Zhou, and the proposer.

The Hyperbolic Tangent in Recursion **April 2001**

1619. *Proposed by Costas Efthimiou, Department of Physics, University of Central Florida, Orlando, FL.*

Consider the real sequences $(a_k)_{k \ge 0}$ that satisfy the recurrence relation

$$\cos a_{n+m} = \frac{\cos a_n + \cos a_m}{1 + \cos a_n \cos a_m}$$

for all nonnegative integers n, m. Such a sequence will be called *minimal* if $0 \le a_k < \pi$ for each k. Determine all minimal sequences.

Solution by Reza Akhlaghi, Prestonsburg Community College, Prestonsburg, KY.

Let $c_k = \frac{1 - \cos a_k}{1 + \cos a_k}$. Then

$$c_{m+n} = \frac{1 - \frac{\cos a_n + \cos a_m}{1 + \cos a_n \cos a_m}}{1 + \frac{\cos a_n + \cos a_m}{1 + \cos a_n \cos a_m}} = \frac{1 - \cos a_n}{1 + \cos a_n} \cdot \frac{1 - \cos a_m}{1 + \cos a_m} = c_n c_m.$$

It follows that $c_n = c_0 c_1^n$. Because $c_0 = c_0^2$, we have $c_0 = 0$ or $c_0 = 1$.

If $c_0 = 0$, then $c_n = 0$ for $n = 0, 1, 2, \ldots$ and it follows that $a_n = 0$ for all n.

If $c_0 = 1$ and $c_1 = 0$, then $a_0 = \pi/2$ and $a_n = 0$ for $n = 1, 2, \ldots$.

If $c_0 = 1$ and $c_1 \neq 0$, then because $c_1 > 0$, we have $c_1 = e^{-2c}$ for some real c. Then $c_n = e^{-2cn}$ and

$$\cos a_n = \frac{1 - e^{-2cn}}{1 + e^{-2cn}} = \frac{e^{cn} - e^{-cn}}{e^{cn} + e^{-cn}} = \tanh(cn).$$

Hence $a_0 = \pi/2$ and $a_n = \arccos(\tanh(cn))$, $n = 1, 2, \ldots$.

Also solved by Michael Bataille (France), Jeffrey Clark, Jim Delany, Charles R. Diminnie, Daniele Donini (Italy), Robert L. Doucette, Ovidiu Furdui, Tom Jager, Howard C. Morris, Markus Neher (Germany), Heinz-Jürgen Seiffert (Germany), Problem Solving Seminar at UAB, Li Zhou, and the proposer.

Medians Weighted on the Side **April 2001**

1620. *Proposed by Mihàly Bencze, Romania.*

In triangle ABC, let $a = BC$, $b = CA$, and $c = AB$. Let m_a, m_b, and m_c be, respectively, the length of the medians from A, B, and C, let s be the semiperimeter, and let R be the circumradius of the triangle. Prove that

$$\max\{am_a,\ bm_b,\ cm_c\} \le sR.$$

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL.*

We prove that $am_a \le sR$. The inequalities $bm_b \le sR$ and $cm_c \le sR$ can be established in a similar way. It is well known that

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{and} \quad R = \frac{a}{2\sin A} = \frac{abc}{4[ABC]},$$

where $[ABC]$ denotes the area of triangle ABC. Using Heron's formula for the area, the inequality $am_a \le sR$ is transformed into

$$2a\sqrt{2b^2 + 2c^2 - a^2} \le s\frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}}.$$

This is equivalent to the inequality

$$b^2c^2(a+b+c) - (2b^2 + 2c^2 - a^2)(a+b-c)(b+c-a)(c+a-b) \ge 0.$$

This last inequality can be obtained by adding the three inequalities:

$$b^2c^2(a+b-c) - b^2(a+b-c)(b+c-a)(c+a-b) = b^2(a+b-c)(a-b)^2 \ge 0,$$

$$b^2c^2(c+a-b) - c^2(a+b-c)(b+c-a)(c+a-b) = c^2(c+a-b)(a-c)^2 \ge 0,$$

and

$$\begin{aligned} &b^2c^2(b+c-a) - (b^2+c^2-a^2)(a+b-c)(b+c-a)(c+a-b) \\ &= (b+c-a)\left[b^2c^2 + (b^2+c^2-a^2)^2 - 2bc(b^2+c^2-a^2)\right] \\ &\ge (b+c-a)\left[2bc|b^2+c^2-a^2| - 2bc(b^2+c^2-a^2)\right] \ge 0. \end{aligned}$$

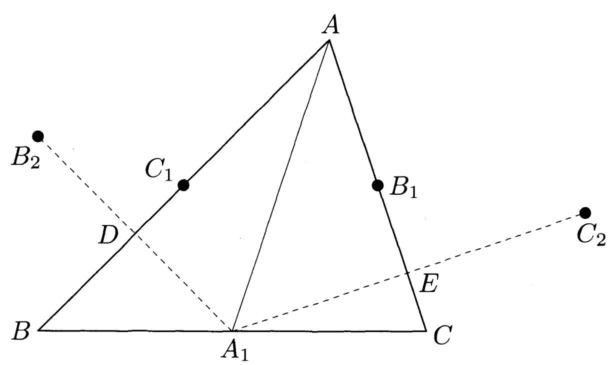
The inequalities also reveal that $am_a = sR$ if and only if triangle ABC is equilateral.

II. *Solution by the proposer.*

Let A_1, B_1, and C_1, respectively, be the midpoints of BC, CA, and AB. Points B_2 and C_2 are chosen so that segment AB is a perpendicular bisector of A_1B_2 and segment AC is a perpendicular bisector of A_1C_2. Let A_1B_2 intersect AB in D and A_1C_2 intersect AC in E. Because $B_1C_2 = B_1A_1$ and $C_1B_2 = C_1A_1$ we have

$$\begin{aligned} 2DE = B_2C_2 &\le C_2B_1 + B_1C_1 + C_1B_2 \\ &= A_1B_1 + B_1C_1 + C_1A_1 = P_{A_1B_1C_1} = \tfrac{1}{2}P_{ABC} = s, \end{aligned} \tag{1}$$

where P_{XYZ} denotes the perimeter of triangle XYZ.



Next note that $\angle A_1DB = \angle A_1EC = 90°$, so quadrilateral ADA_1E is cyclic. Thus $\angle DA_1E = 180° - \angle A$ and $\angle A_1AE = \angle A_1DE$. Applying the Law of Sines in triangle A_1DE and then in triangle A_1AE, we have

$$\frac{DE}{\sin A} = \frac{DE}{\sin(DA_1E)} = \frac{A_1E}{\sin(A_1DE)} = \frac{A_1E}{\sin(A_1AE)} = AA_1 = m_a.$$

Using (1) we have

$$s \ge 2DE = 2m_a \sin A = 2m_a\frac{a}{2R},$$

and it follows that $am_a \le sR$. A similar argument shows that $bm_b \le sR$ and $cm_c \le sR$, establishing the desired inequality.

Also solved by Herb Bailey, Daniele Donini (Italy), and Volkhard Schindler (Germany). There was one incorrect submission.

Fibonacci Numbers from Binomial Coefficients **April 2001**

1621. *Proposed by Donald Knuth, Stanford University, Stanford, CA.*

Prove that if *a*, *b*, and *n* are arbitrary nonnegative integers, then the sum

$$\sum_{k=-\infty}^{\infty}\left(\binom{n}{a+5k} - \binom{n}{b+5k}\right)$$

is a Fibonacci number or the negative of a Fibonacci number.

Solution by Michael D. Hirschhorn, The University of New South Wales, Sydney, Australia.

Let

$$S = S(n, a) = \sum_{k=-\infty}^{\infty} \binom{n}{a+5k}.$$

When $n = 0$ or $n = 1$, S takes on only the values 0 and 1. For $n \geq 2$, S takes on only three values, which we denote by x_n, y_n, and z_n, with $x_n > y_n > z_n$. When $n = 2k$ is even, S assumes the value x_n when the sum contains the term $\binom{2k}{k}$, the value y_n when the sum contains $\binom{2k}{k-1}$ or $\binom{2k}{k+1}$, and the value z_n when the sum contains the term $\binom{2k}{k-2}$ or $\binom{2k}{k+2}$. When $n = 2k+1$ is odd, S takes on the value x_n when the sum contains the term $\binom{2k+1}{k}$ or $\binom{2k+1}{k+1}$, the value y_n when the sum contains $\binom{2k+1}{k-1}$ or $\binom{2k+1}{k+2}$, and the value z_n when the sum contains the term $\binom{2k+1}{k-2}$. It follows that for n even,

$$x_n = 2x_{n-1}, \qquad y_n = x_{n-1} + y_{n-1}, \qquad z_n = y_{n-1} + z_{n-1},$$

while for n odd,

$$x_n = x_{n-1} + y_{n-1}, \qquad y_n = y_{n-1} + z_{n-1}, \qquad z_n = 2z_{n-1}.$$

With these results it can be shown by induction that for n odd,

$$\begin{aligned} x_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots - 2F_n + F_{n+1}, \\ y_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots - 2F_n + F_{n-1}, \\ z_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots - 2F_n, \end{aligned}$$

where F_n is the n^{th} Fibonacci number, and for n even,

$$\begin{aligned} x_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots + 2F_n, \\ y_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots + 2F_n - F_{n-1}, \\ z_n &= 2^{n-1}F_2 - 2^{n-2}F_3 - \cdots + 2F_n - F_{n+1}. \end{aligned}$$

It follows that for nonnegative integers n, a, and b, $S(n, a) - S(n, b)$ is a Fibonacci number or the negative of a Fibonacci number.

It is possible to give other formulations of the solution. Several papers have been written on this subject. See, for example,

1. George E. Andrews, Some formulae for the Fibonacci sequence with generalizations, *The Fibonacci Quarterly*, **7** (1969), 113–130.
2. Hansraj Gupta, The Andrews formula for Fibonacci numbers, *The Fibonacci Quarterly*, **16** (1978), 552–555.
3. Michael D. Hirschhorn, The Andrews formula for Fibonacci numbers, *The Fibonacci Quarterly*, **19** (1981), 1–2.

Also solved by Jany C. Binz (Switzerland), David M. Bloom, Fred T. Howard, Reiner Martin, Rob Pratt, Heinz-Jürgen Seiffert (Germany), Albert Stadler (Switzerland), Michael Woltermann, and the proposer.

A Positive Definite Infimum **April 2001**

1622. *Proposed by Götz Trenkler, Dortmund, Germany.*

On the family $\mathcal{P}_n$ of $n \times n$ positive definite matrices, define the partial order $\leq_L$ by

$$M \leq_L N \quad \text{if and only if} \quad N - M \quad \text{is positive semi-definite.}$$

(This is a Loewner ordering on $\mathcal{P}_n$.) Find a matrix C in $\mathcal{P}_n$ that is the greatest lower bound, with respect to $\leq_L$, for the set

$$\{(A + A^{-1})^2 + (B + B^{-1})^2 : A \in \mathcal{P}_n,\ B = I_n - A \in \mathcal{P}_n\}.$$

Solution by Tom Jager, Calvin College, Grand Rapids, MI.

The greatest lower bound is $C = \frac{25}{2} I_n$ and is achieved when $A = \frac{1}{2} I_n$. First observe that if $A = \frac{1}{2} I_n$, then $B = A$ and

$$(A + A^{-1})^2 + (B + B^{-1})^2 = \left(\frac{5}{2} I_n\right)^2 + \left(\frac{5}{2} I_n\right)^2 = \frac{25}{2} I_n.$$

Now let $A \in P_n$ and $B = I_n - A \in P_n$. Because A and B are Hermitian and positive definite, there is a unitary matrix U so that $U^* A U = D(c_1, c_2, \ldots, c_n) = D(c_i)$ is a diagonal matrix with real diagonal entries c_i satisfying $0 < c_i < 1$. Then

$$\begin{aligned} T &= (A + A^{-1})^2 + (B + B^{-1})^2 \\ &= U\left[(D(c_i) + D(1/c_i))^2 + (D(1 - c_i) + D(1/(1 - c_i)))^2\right] U^* \\ &= U D\left(\left(c_i + \frac{1}{c_i}\right)^2 + \left(1 - c_i + \frac{1}{1 - c_i}\right)^2\right) U^*. \end{aligned}$$

Define f by $f(c) = (c + 1/c)^2 + (1 - c + 1/(1 - c))^2$, $0 < c < 1$. On this interval f assumes its minimum value of $25/2$ when $c = 1/2$. Thus, $U^* T U = D(a_i)$, where each $a_i \geq 25/2$. Thus $T - \frac{25}{2} I_n$ is positive semi-definite and $\frac{25}{2} I_n \leq_L T$.

Also solved by the proposer.

Answers

Solutions to the Quickies from page 146.

A919. By the Law of Cosines,

$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2AB \cdot BD \cos(\angle ABC + 60^\circ) \\ &= AB^2 + BC^2 - 2AB \cdot BC\,(\cos(\angle ABC)\cos 60^\circ - \sin(\angle ABC)\sin 60^\circ) \\ &= AB^2 + AC^2 + \sqrt{3} AB \cdot AC. \end{aligned}$$

It follows that AB, AC, and AD cannot all be rational.

A920. Under the transformation $(x, y) \longrightarrow (bx/a, y)$, the parallelogram is transformed to another parallelogram and the ellipse is transformed to an inscribed circle of radius b. The circle is tangent to the image parallelogram at the points $(b\cos u, b\sin u)$ and $(b\cos v, b\sin v)$. Draw radii to the four point of tangency. The result is two quadrilaterals of area $b^2\tan(\frac{v-u}{2})$ and two of area $b^2\tan(\frac{\pi - v + u}{2})$, for a total area of $4b^2\csc(v - u)$. Because ratios of areas are preserved under the transformation, the desired area is $4ab\csc(v - u)$.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Ascher, Marcia, The Kolam tradition, *American Scientist* 90 (January-February 2002) 56–63; `http://americanscientist.org/articles/02articles/Ascher.html`.

The women of Tamil Nadu (in southeastern India) daily decorate their thresholds with elaborate symmetrical figures, called *kolam*, made by sprinkling rice powder. Families of these figures have been analyzed by computer scientists as picture languages. The families can be described formally as context-free deterministic Lindenmayer systems, that is, symbol replacement systems in which replacement rules are applied in parallel at each step. A symbol string can be converted to a graphic like a kolam figure by interpreting each symbol as a motion command to a graphics turtle. A group of researchers in Madras was inspired by the process of drawing kolam figures to work on *array grammars*, replacement systems in which the items being replaced are subarrays of symbols rather than strings of symbols. These grammars can describe how kolam figures reproduce units of a pattern, and the algorithmic nature of the grammars systematizes the procedures and techniques behind kolam drawing.

Kosko, Bart, How many blonds [sic] mess up a Nash equilibrium, *Los Angeles Times* (13 February 2002), `http://www.latimes.com/news/printededition/california/la-000011031feb13.story`. Abbott, Russ, and Morrissey, Brendan, Letters, *Los Angeles Times* (18 February 2002), `http://www.latimes.com/news/printededition/california/la-000012467feb18.story`.

Kosko contends that the film "A Beautiful Mind" gives an incorrect example to explain what a Nash equilibrium is. The Nash character claims that the optimal strategy for him and others in a bar who desire a blonde there is to pursue the accompanying brunettes, since the men can't all have the blonde. This can't be an equilibrium because each man will switch to the blonde if the others pursue brunettes. The accompanying letters offer a more typical example of Nash equilibrium (in marketing) and a critique of an analogy used by Kosko.

Netzer, Greg, A better golf ball, *New York Times Magazine* (9 December 2001) 58, 60; `http://www.nytimes.com/2001/12/09/magazine/09GOLFBALL.html`.

By the time you read this, the new Callaway HX golf ball should be in pro shops. The dimples on golf balls reduce drag on the ball; on current balls, dimples cover 65–75% of the area of the ball. In 1999 a researcher at Callaway found a pattern of dimples that covered 86% (hmmm... did he consult N.J.A. Sloane's catalog of sphere coverings at `http://www.research.att.com/~njas/icosahedral.codes/index.html`?). The new ball, however, has no circular dimples but instead is covered with ridges surrounding pentagonal and hexagonal regions (quick: if pentagons and hexagons tessellate the sphere edge-to-edge, exactly how many are pentagons?*). The new ball meets U.S.G.A. specifications and is claimed (of course) to sail farther.

*HINTS: 1. The question claims that the answer is the same for all tessellations; assuming that the claim is correct, you can deduce the answer by considering a familiar polyhedron that has no hexagons. 2. For a proof, count the edges two ways and apply Euler's relation among the numbers of vertices, edges, and polygons of a polyhedron.

Salsburg, David, *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century*, W.H. Freeman, 2001; xi + 340 pp, $23.95.

I teach statistics every semester, so I was eager to dip into this book and see if it would provide good supplementary reading for me or even be suitable reading for students to complement the technique orientation of the usual textbook. The book takes the interesting tactic of describing the "statistical revolution in twentieth-century science"—from deterministic models of reality to statistical ones—"in terms of statisticians involved in that revolution." The book is enriched by the fact that the author (a statistician formerly in pharmaceutical research and a Fellow of the American Statistical Society) knows or knew many of those individuals. I found the book easy reading, learned some things I had not known, and found a few gems to enliven classes; but the fare was lighter than I would have preferred, and not because there is not a formula nor even a sigma of symbolic notation anywhere in the book. I think I was hoping for less biography and more subtle insights. Perhaps others too will pine for a bit more depth. An example of thin exposition is on p. 289, where two-thirds of a page is devoted to Efron's bootstrap method; but we are not told what it is, what it is for, nor even what the "standard methods" are that it is "equivalent to." We do learn that it is "based on two simple applications of the Glivenko-Cantelli lemma" (that part was news to me). Overall, I had the impression of having taken a whirlwind tour of sights without being told by the guide just what some of them were or why we toured them.

Peterson, Ivars, *Mathematical Treks: From Surreal Numbers to Magic Circles*, MAA, 2002; x + 170 pp, $24.95 (P) ($19.95 to MAA members). ISBN 0–88385–537–2.

Science News's debut on the Web allowed the weekly magazine to post there more material than could fit in its printed edition, including a weekly column on mathematics by Ivars Peterson. His columns are available at the MAA Website, and this book collects updated versions of the columns of the first year and a half. They are crisp vignettes, complete with references, on a vast variety of mathematical topics; though shorter, they carry on the tradition of Martin Gardner's former "Mathematical Games" column in *Scientific American.*

Yandell, Benjamin H., *The Honors Class: Hilbert's Problems and Their Solvers*, A K Peters, 2002; ix + 486 pp, $39. ISBN 1–56881–141–1.

Hermann Weyl wrote that by solving one of the famous problems identified by David Hilbert in his 1900 address to the Second International Congress of Mathematicians, a mathematician "passed on to the honors class of the mathematical community." It is a small class. This book follows the "progress of the problems. Who worked on them? Who solved them? How were they solved? Which have not been solved? And what developed in twentieth century mathematics that Hilbert left out?" The book is intended for both mathematical and nonmathematical readers; the latter are advised that when they come to something that they don't understand, they should just keep reading and "pretend you are reading *Moby Dick* and you've come to another section on whaling." There are endnotes (citing sources for quotations), photographs, and even a few mathematical expressions; and Hilbert's address (translated) is reprinted. The result is a book that will delight both of the intended audiences and provide an overview of important contributions of twentieth-century mathematics.

Peterson, Ivars, *Fragments of Infinity: A Kaleidoscope of Math and Art*, Wiley, 2001; vii + 232 pp, $29.95. ISBN 0–471–16558–1.

This book is devoted to art inspired by mathematics; it contains more lustrous photos (including eight pages of color plates), figures, and illustrations than pages. It touches on origami, quasicrystals, Costa surfaces, Escher, and variations on Möbius strips, among other topics. You may wish that your nonmathematical friends would read Yandell's *The Honors Class* (above), but this is the book that they won't stop looking through.

NEWS AND LETTERS

30th United States of America Mathematical Olympiad

May 1, 2001

edited by Titu Andreescu and Zuming Feng

Problems

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.
2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.
3. Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

   Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

4. Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.
5. Let S be a set of integers (not necessarily positive) such that
   (a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;
   (b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

   Prove that S is the set of all integers.
6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solutions

Note: Each year the MAA publishes a book with multiple solutions, references, and results for both the USA Mathematical Olympiad and the International Mathematical Olympiad. The first book in that series, *USA and International Mathematical Olympiads 2000*, is available now from the MAA, and the second, *USA and Inter-*

national Mathematical Olympiads 2001 will be published after the 42nd International Mathematical Olympiad this summer.

1. The smallest such n is 23.

   We first show that $n = 22$ cannot be achieved.

   Assume that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of $6 + 5 + 5 + 5 + 2 + 1 = 24$, a contradiction.

   Next, assume that no color occurs four times; this forces at least four colors to occur three times. In particular, there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of $6 + 5 + 5 + 3 + 3 + 1 = 23$, again a contradiction.

   We now give a construction for $n = 23$. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

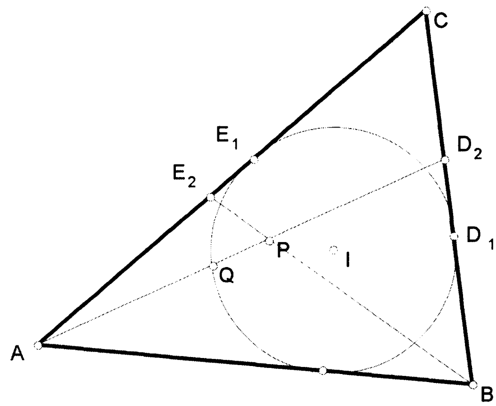
1	3	4	5	6	7
1	8	9	10	11	12
1	13	14	15	16	17
2	3	8	13	18	19
2	4	9	14	20	21
2	5	10	15	22	23
6	11	16	18	20	22
7	12	17	19	21	23

2. The key observation is that segment D_1Q is a diameter of circle ω. Let I be the center of circle ω, that is, I is the incenter of triangle ABC. Extend segment D_1I through I to intersect circle ω again at Q', and extend segment AQ' through Q' to intersect segment BC at D'. We show that $D_2 = D'$, which in turn implies that $Q = Q'$, that is, D_1Q is a diameter of ω.

   Let ℓ be the line tangent to circle ω at Q', and let ℓ intersect segments AB and AC at B_1 and C_1, respectively. Then ω is an excircle of triangle AB_1C_1. Let $\mathbf{H}_1$ denote the dilation with center A and ratio AD'/AQ'. Since $\ell \perp D_1Q'$ and $BC \perp D_1Q$, $\ell \parallel BC$. Hence, $AB/AB_1 = AC/AC_1 = AD'/AQ'$. Thus, $\mathbf{H}_1(Q') = D'$, $\mathbf{H}_1(B_1) = B$, and $\mathbf{H}_1(C_1) = C$. It also follows that the excircle Ω of triangle ABC opposite vertex A is tangent to side BC at D'.

   We compute BD'. Let X and Y denote the points of tangency of circle Ω with rays AB and AC, respectively. Then by equal tangents, $AX = AY$, $BD' = BX$, and $D'C = YC$. Hence,

$$\begin{aligned} AX = AY &= \tfrac{1}{2}(AX + AY) \\ &= \tfrac{1}{2}(AB + BX + YC + CA) = \tfrac{1}{2}(AB + BC + CA). \end{aligned}$$



It follows that $BD' = BX = AX - AB = \frac{1}{2}(BC + CA - AB)$. It is well known that $CD_1 = \frac{1}{2}(BC + CA - AB)$. Hence $BD' = CD_1$. Thus, $BD_2 = BD_1 - D_2D_1 = D_2C - D_2D_1 = D_1C = BD'$, that is, $D' = D_2$, as claimed.

Now we prove our main result. Let M_1 and M_2 be the midpoints of segments BC and CA, respectively. Then M_1 is also the midpoint of segment D_1D_2, from which it follows that IM_1 is the midline of triangle D_1QD_2. Hence,

$$QD_2 = 2IM_1 \tag{1}$$

and $AD_2 \parallel M_1I$. Similarly, we can prove that $BE_2 \parallel M_2I$.

Let G be the centroid of triangle ABC. Thus, segments AM_1 and BM_2 intersect at G. Define transformation $\mathbf{H}_2$ as the **dilation** with center G and ratio $-1/2$. Then $\mathbf{H}_2(A) = M_1$ and $\mathbf{H}_2(B) = M_2$. Under the dilation, parallel lines go to parallel lines and the intersection of two lines goes to the intersection of their images. Since $AD_2 \parallel M_1I$ and $BE_2 \parallel M_2I$, $\mathbf{H}$ maps lines AD_2 and BE_2 to lines M_1I and M_2I, respectively. It also follows that $\mathbf{H}_2(P) = I$ and that $IM_1/AP = GM_1/AG = 1/2$, or $AP = 2IM_1$. Combining the last equality with (1) yields $AQ = AP - QP = 2IM_1 - QP = QD_2 - QP = PD_2$, as desired.

3. From the condition, at least one of a, b, and c does not exceed 1, say $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.$$

To obtain equality, we have $a(b + c) = bc(1 - a) = 0$. If $a = 1$, then $b + c = 0$ or $b = c = 0$, which contradicts the given condition $a^2 + b^2 + c^2 + abc = 4$. Hence $1 - a \neq 0$ and only one of b and c is 0. Without loss of generality, say $b = 0$. Therefore $b + c > 0$ and $a = 0$. Plugging $a = b = 0$ back into the given condition gives $c = 2$. By permutation, the lower bound holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Now we prove the upper bound. Let us note that at least two of the three numbers a, b, and c are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are b and c. Then we have $(1 - b)(1 - c) \geq 0$.

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \geq 2bc$ imply $a^2 + 2bc + abc \leq 4$,or $bc(2 + a) \leq 4 - a^2$. Dividing both sides of the last inequality by $2 + a$ yields $bc \leq 2 - a$.

It follows that

$$\begin{aligned} ab + bc + ac - abc &\le ab + 2 - a + ac(1-b) \\ &= 2 - a(1 + bc - b - c) = 2 - a(1-b)(1-c) \le 2, \end{aligned}$$

as desired.

The last equality holds if and only if $b = c$ and $a(1-b)(1-c) = 0$. Hence, equality for the upper bound holds if and only if (a, b, c) is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$, and $(\sqrt{2}, \sqrt{2}, 0)$.

4. By the Cauchy-Schwarz Inequality,

$$\sqrt{PB^2 + PC^2}\sqrt{AB^2 + AC^2} \ge PB \cdot AC + PC \cdot AB.$$

Applying the Generalized Ptolemy's Inequality to quadrilateral $ABPC$ yields

$$PB \cdot AC + PC \cdot AB \ge PA \cdot BC.$$

Because PA is the longest side of an obtuse triangle with side lengths PA, PB, PC, we have $PA > \sqrt{PB^2 + PC^2}$ and hence

$$PA \cdot BC \ge \sqrt{PB^2 + PC^2} \cdot BC.$$

Combining these three inequalities yields $\sqrt{AB^2 + AC^2} > BC$, implying that angle BAC is acute.

5. In the solution below we use the expression S *is stable under* $x \mapsto f(x)$ to mean that if t belongs to S, then $f(t)$ also belongs to S. If $c, d \in S$, then by condition (b), S is stable under $x \mapsto c^2 - x$ and $x \mapsto d^2 - x$. Hence, it is stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly, S is stable under $x \mapsto x + (d^2 - c^2)$. Hence, S is stable under $x \mapsto x + n$ and $x \mapsto x - n$, whenever n is an integer linear combination of finitely many numbers in $T = \{c^2 - d^2 \mid c, d \in S\}$.

By condition (a), $S \neq \emptyset$ and hence $T \neq \emptyset$ as well. For the sake of contradiction, assume that some p divides every element in T. By condition (b), $a^2 - a, b^2 - b \in S$. Therefore, p divides $a^2 - b^2$, $x_1 = (a^2 - a)^2 - a^2$, and $x_2 = (b^2 - b)^2 - b^2$. Because $\gcd(a, b) = 1$, both $\gcd(a^2 - b^2, a^3)$ and $\gcd(a^2 - b^2, b^3)$ equal 1, so p does not divide a^3 or b^3. But p does divide $x_1 = a^3(a-2)$ and $x_2 = b^3(b-2)$, so it must divide $a - 2$ and $b - 2$. Because $\gcd(a-2, b-2) = 1$ by condition (a), this implies $p \mid 1$, a contradiction. Therefore our original assumption was false, and no such p exists.

It follows that $T \neq \{0\}$. Let x be an arbitrary nonzero element of T. For each prime divisor of x, there exists an element in T which is not divisible by that prime. The set A consisting of x and each of these elements is finite. By construction, $m = \gcd\{y \mid y \in A\} = 1$, and m can be written as an integer linear combination of finitely many elements in A and hence in T. Therefore, S is stable under $x \mapsto x + 1$ and $x \mapsto x - 1$. Because S is nonempty, it follows that S is the set of all integers.

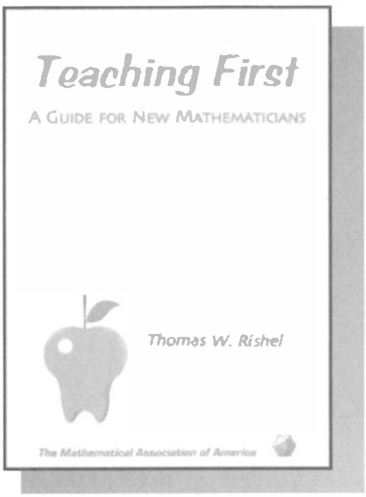
6. Let A, B be arbitrary distinct points, and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and FE intersect at G. Let ℓ be the line through G perpendicular to line ED. Then A, F, E and B, C, D are symmetric to each other, respectively, with respect to line ℓ. Hence triangles CEG and DFG share the same incenter, that is, $c + e = d + f$; triangles ACE and BDF share the same incenter, that is, $a + c + e = b + d + f$. Therefore, $a = b$, and we are done.

The Mathematical Association of America

Teaching First: A Guide for New Mathematicians

Thomas W. Rishel

Series: MAA Notes



In this volume Thomas Rishel draws on his nearly forty years of teaching experience to address the "nuts and bolts" issues of teaching college mathematics. This book is written for the mathematics TA or young faculty member who may be wondering just where and how to start. Rishel opens the readers' eyes to pitfalls they may never have considered, and offers advice for balancing an obligation "to the student" with an obligation "to mathematics." Throughout, he provides answers to seemingly daunting questions shared by most new TAs, such as how to keep a classroom active and lively; how to prepare writing assignments, tests, and quizzes; how exactly to write a letter of recommendation; and how to pace, minute by minute, the "mathematical talks" one will be called upon to give.

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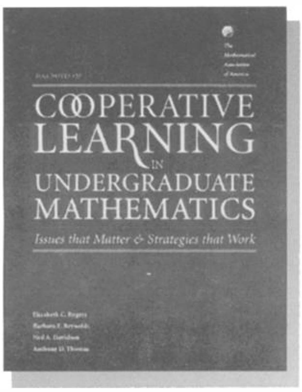
Cooperative Learning in Undergraduate Mathematics: Issues that Matter and Strategies that Work



Elizabeth C. Rogers, Barbara E. Reynolds,
Neil A. Davidson, and Anthony D. Thomas, Editors

Series: MAA Notes

This volume offers practical suggestions and strategies both for instructors who are already using cooperative learning in their classes, and for those who are thinking about implementing it. The authors are widely experienced with bringing cooperative learning into the undergraduate mathematics classroom. In addition they draw on the experiences of colleagues who responded to a survey about cooperative learning which was conducted in 1996-97 for Project CLUME (Cooperative Learning in Undergraduate Mathematics Education).

The volume discusses many of the practical implementation issues involved in creating a cooperative learning environment:

- how to develop a positive social climate, form groups and prevent or resolve difficulties within and among the groups.
- what are some of the cooperative strategies (with specific examples for a variety of courses) that can be used in courses ranging from lower-division, to calculus, to upper division mathematics courses.
- what are some of the critical and sensitive issues of assessing individual learning in the context of a cooperative learning environment.
- how do theories about the nature of mathematics content relate to the views of the instructor in helping students learn that content.

The authors present powerful applications of learning theory that illustrate how readers might construct cooperative learning activities to harmonize with their own beliefs about the nature of mathematics and how mathematics is learned.

In writing this volume the authors analyzed and compared the distinctive approaches they were using at their various institutions. Fundamental differences in their approaches to cooperative learning emerged. For example, choosing Davidson's guided-discovery model over a constructivist model based on Dubinsky's action-process-object-schema (APOS) theory affects one's choice of activities. These and related distinctions are explored.

A selected bibliography provides a number of the major references available in the field of cooperative learning in mathematics education. To make this bibliography easier to use, it has been arranged in two sections. The first section includes references cited in the text and some sources for further reading. The second section lists a selection (far from complete) of textbooks and course materials that work well in a cooperative classroom for undergraduate mathematics students.

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